# String Connections and Chern-Simons Theory

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**Abstract**

We present a finite-dimensional and smooth formulation of string structures on spin bundles. It uses trivializations of the Chern-Simons 2-gerbe associated to this bundle. Our formulation is particularly suitable to deal with string connections. We prove that every string structure admits a string connection and that the possible choices form a contractible space. We also provide a new relation between string connections and 3-forms on the base manifold.

**Keywords**: geometric string structure, string connection, 2-gerbe, Chern-Simons theory.

# Introduction

This article is concerned with a smooth manifold $M$, a principal spin bundle $P$ over $M$, and an additional structure on $P$ called "string structure". From a purely topological point of view, a string structure on $P$ is a certain cohomology class $\xi \in \mathrm{H}^3(P, \mathbb{Z})$.

We prove that a string structure on $P$ can equivalently be understood as a trivialization of a geometrical object over $M$, namely of a bundle 2-gerbe $\mathbb{CS}_P$. This bundle 2-gerbe is canonically associated to $P$ and plays in important role in classical Chern-Simons theory. Our aim is to use this new formulation as a suitable setup to study string connections.

A string connection is an additional structure for a connection $A$ on $P$ and a string structure on $P$. The combination of a string structure and a string connection is also called a "geometric string structure" on the pair

$(P, A)$. We give a new and simple definition of a string connection in terms of connections on bundle 2-gerbes, and show that it is equivalent to the original notion introduced by Stolz and Teichner [ST04]. We provide the following list of new results on string connections:

- Geometric string structures form a 2-category, whose set of isomorphism classes of objects is parameterized by the degree three differential cohomology of $M$.
- Canonically associated to every string connection is a 3-form on $M$, whose pullback to $P$ differs from the Chern-Simons 3-form by a closed 3-form with integral periods.
- For every string structure on $P$ and every connection $A$ there exists a string connection. The set of possible string connections is an affine space.

Back in the eighties, string structures emerged from the study of two-dimensional supersymmetric field theories by Killingback [Kil87] and Witten [Wit86]. In these applications, the principal spin bundle $P$ is a spin structure on an oriented Riemannian manifold $M$. In this case one also speaks about a string structure on $M$ rather than on $P$. More recently, an interesting relation between string structures on $M$, metrics of positive Ricci curvature and the Witten genus has been conjectured [Sto96].

A first mathematical framework for string structures has been developed by McLaughlin [McL92]: a string structure on $P$ is a lift of the structure group of the "looped" bundle $LP$ over $LM$ from $L\mathrm{Spin}(n)$ to its basic central extension. Another formulation has been proposed by Stolz and Teichner [ST04]: a string structure on $P$ is a lift of the structure group of $P$ from $\mathrm{Spin}(n)$ to a certain three-connected central extension: the string group. This group is *not* a Lie group, but can be realized by an infinite-dimensional Fréchet Lie 2-group [BCSS07].

Equivalence classes of Stolz-Teichner string structures on $P$ are in bijection to certain classes $\xi \in \mathrm{H}^3(\mathrm{Spin}(n), \mathbb{Z})$; this leads to the formulation we have used above.

One common aspect of McLaughlin's and the Stolz-Teichner approach is that both reveal string structures as objects in a category rather than elements of a set. This is essential if one wants to equip string structures with additional data like string connections. Another aspect is that – although the underlying setup (the bundle $P$ and the manifold $M$) is finite-dimensional – both approaches involve infinite-dimensional structures!

The main innovation of our new formulation of a string structure is that it remains in the category of finite-dimensional, smooth manifolds. This becomes possible due to recent work of Gawędzki-Reis [GR02] and Meinrenken [Mei02] on the "basic gerbe" over a simple, compact Lie group, which enters the construction of the Chern-Simons 2-gerbe $\mathbb{CS}_P$. A second novelty is that our formulation exhibits string structures as objects in a 2-category rather than in a category: we thus uncover one further layer of structure.

String connections have been introduced by Stolz and Teichner as "trivializations" of a certain Chern-Simons theory [ST04]. Our new definition of a string connection has the advantage that it can be expressed as a couple of real-valued differential forms of degrees one and two. Our approach also shows that string connections can systematically be understood as an application of the theory of connections on bundle 2-gerbes. In the proofs of our results we use and extend this existing theory.

Our results on string connections make progress in two aspects. The first concerns the relation between string structures and 3-forms on $M$, which became apparent in Redden's thesis in the presence of a Riemannian metric on $M$ [Red06]. Here we could identify the 3-form as a genuine feature of a string connection. The second aspect is that we settle the existence of string connections and the homotopy type of the space of string connections. These results verify a conjecture of Stolz and Teichner.

This article is organized in the following way. In the next section we present a detailed overview on our results. Sections 2 and 3 provide definitions and outline the proofs. In Section 4 we provide some background in bundle gerbe theory. All technical issues are collected in Section 5.

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# 1 Summary of the Results

We explain in Section 1.1 how string structures can be understood as trivializations of the Chern-Simons 2-gerbe. In Section 1.2 we upgrade to a setup with connections, and explain the relation between string connections and Chern-Simons theory. In Section 1.3 we present our results on string connections.

## 1.1 String Structures as Trivializations

Let us start with the topological definition of a string structure and recall one result. Throughout this article $M$ is a smooth manifold and $n$ is an integer, $n > 4$. Note that $\mathrm{Spin}(n)$ is simple and simply-connected, and of course compact. In particular, there is a preferred generator of $\mathrm{H}^3(\mathrm{Spin}(n), \mathbb{Z}) \cong \mathbb{Z}$.

**Definition 1.1.1** ([Red06], Def. 6.4.2)**.** *Let $P$ be a principal $\mathrm{Spin}(n)$-bundle over $M$. A string structure on $P$ is a class $\xi \in \mathrm{H}^3(P, \mathbb{Z})$, such that for every point $p \in P$ the associated inclusion*

$$\iota_p : \mathrm{Spin}(n) \longrightarrow P : g \longmapsto p.g$$

*pulls $\xi$ back to the generator of $\mathrm{H}^3(\mathrm{Spin}(n), \mathbb{Z})$.*

The string structures from our Definition 1.1.1 are in bijection to equivalence classes of string structures in the sense of Stolz and Teichner [ST04]. In their formulation, a string structure is a lift of the structure group of $P$ from $\mathrm{Spin}(n)$ to a certain central extension

$$1 \longrightarrow B\mathrm{U}(1) \longrightarrow \mathrm{String}(n) \longrightarrow \mathrm{Spin}(n) \longrightarrow 1.$$

Such a lift can be seen as a $B\mathrm{U}(1)$-bundle over $P$; such bundles have characteristic classes $\xi \in \mathrm{H}^3(P, \mathbb{Z})$. Properties of the string group imply the condition on $\xi$ from Definition 1.1.1. We refer to [Red06], Prop. 6.4.3 for a more detailed discussion.

One can ask whether a principal $\mathrm{Spin}(n)$-bundle $P$ admits string structures, and if it does, how many. Associated to $P$ is a class in $\mathrm{H}^4(M, \mathbb{Z})$ which is – when multiplied by two – the first Pontryagin class of the underlying $\mathrm{SO}(n)$-bundle. Therefore it is denoted by $\frac{1}{2}p_1(P)$.

**Theorem 1.1.2** ([ST04], Sec. 5)**.** *Let $\pi : P \longrightarrow M$ be a principal $\mathrm{Spin}(n)$-bundle over $M$.*

*(a) $P$ admits string structures if and only if $\frac{1}{2}p_1(P) = 0$.*

*(b) If $P$ admits string structures, the possible choices form a torsor over the group $\mathrm{H}^3(M, \mathbb{Z})$, where the action of $\eta \in \mathrm{H}^3(M, \mathbb{Z})$ takes a string structure $\xi$ to the string structure $\xi + \pi^*\eta$.*

Next we report the first results of this article. It is a rather trivial observation that the group $\mathrm{H}^4(M, \mathbb{Z})$ classifies geometrical objects over $M$ called *bundle 2-gerbes*. In this article, bundle 2-gerbes are defined internal to the

category of smooth, finite-dimensional manifolds (see Definition 2.1.1). The before-mentioned classification result for bundle 2-gerbes implies that there is an isomorphism class of bundle 2-gerbes over $M$ consisting of bundle 2-gerbes with a characteristic class equal to $\frac{1}{2}p_1(P)$. Our first result is the observation that this isomorphism class has a canonical representative.

**Theorem 1.1.3.** *Let $P$ be a principal $\mathrm{Spin}(n)$-bundle over $M$. Then, there exists a canonical bundle 2-gerbe over $M$ whose characteristic class is $\frac{1}{2}p_1(M)$. This bundle 2-gerbe is called the Chern-Simons 2-gerbe associated to $P$, and is denoted $\mathbb{CS}_P$.*

An explicit construction of the Chern-Simons 2-gerbe is given in Section 2.1; it comes from a more general construction due to Carey, Johnson, Murray, Stevenson and Wang [CJM+05].

We remark that there also exist geometrical constructions of at least $p_1(P)$ in other setups: one by Brylinski and McLaughlin in terms of sheaves of 2-groupoids [BM92] and one by Stevenson [Ste04], in terms of infinite-dimensional bundle 2-gerbes.

In order to detect whether $\frac{1}{2}p_1(P)$ vanishes or not we use a geometrical criterion for the vanishing of the characteristic class of a bundle 2-gerbe, called *trivialization*. Trivializations of a fixed bundle 2-gerbe $\mathbb{G}$ form a 2-groupoid $\mathcal{T}\!riv(\mathbb{G})$. We show

**Theorem 1.1.4.** *The bundle $P$ admits string structures if and only if the Chern-Simons 2-gerbe $\mathbb{CS}_P$ has a trivialization. If $P$ admits string structures, there is a canonical bijection*

$$\left\{ \begin{array}{c} \textit{Isomorphism classes of} \\ \textit{trivializations of } \mathbb{CS}_P \end{array} \right\} \cong \{ \textit{ String structures on } P \,\} \,.$$

The first part follows directly from a general result of Stevenson [Ste04], see Lemma 2.2.2. The second part are Propositions 2.2.3 and 2.2.6 in Section 2.2: there we note that both sides are torsors over $\mathrm{H}^3(M,\mathbb{Z})$ and specify a canonical, equivariant map.

In the following parts of this article we take the point of view that a string structure on $P$ *is* a trivialization of the associated Chern-Simons 2-gerbe $\mathbb{CS}_P$. As a consequence, string structures are not just a set as in Definition 1.1.1, they inherit the structure of trivializations of bundle 2-gerbes:

**Corollary 1.1.5.** *String structures on a principal $\mathrm{Spin}(n)$-bundle $P$ over $M$ form a 2-groupoid $\mathcal{T}\!riv(\mathbb{CS}_P)$.*

## 1.2 String Connections as Connections on Trivializations

The main objective of this article is to show that understanding string structures as trivializations of the Chern-Simons 2-gerbe has many advantages, in particular when additional, differential-geometric structures become involved.

An example for such a differential-geometric structure is a *string connection*. Our first goal is to give a simple definition of a string connection. We shall use existing definitions and results on connections on bundle 2-gerbes. Basically, a connection on a bundle 2-gerbe is the same type of additional structure an ordinary connection is for an ordinary principal bundle. For example, every connection on a bundle 2-gerbe has a curvature, which is a closed 4-form on $M$ that represents the characteristic class of the underlying bundle 2-gerbe in the real cohomology of $M$.

**Theorem 1.2.1.** *Suppose $A$ is a connection on a principal* $\mathrm{Spin}(n)$*-bundle $P$ over $M$. Then, the Chern-Simons 2-gerbe $\mathbb{CS}_P$ comes equipped with a canonical connection $\nabla_A$. The curvature of $\nabla_A$ is one half of the Pontryagin 4-form associated to $A$.*

We give an explicit construction of the connection $\nabla_A$, based on results about connections on multiplicative bundle gerbes [Wal08]. It involves the Chern-Simons 3-form $TP(A) \in \Omega^3(P)$ which is fundamental in Chern-Simons theory [CS74]; hence the name of the 2-gerbe $\mathbb{CS}_P$. The curvature of $\nabla_A$ is calculated in Lemma 3.1.2.

Now that the Chern-Simons 2-gerbe $\mathbb{CS}_P$ carries a connection $\nabla_A$, one can consider trivializations of $\mathbb{CS}_P$ that preserve the connection $\nabla_A$ in an appropriate way. However, the term "preserve" is not accurate: it is a standard fact in higher gauge theory that for a morphism between two objects with connections, being "connection-preserving" is *structure*, not property. So we better speak of *trivializations with compatible connection*. Generally, if $\mathbb{G}$ is a bundle 2-gerbe with connection $\nabla$, there is a 2-groupoid $\mathcal{T}riv(\mathbb{G}, \nabla)$ of trivializations of $G$ with connection compatible with $\nabla$.

We propose the following

**Definition 1.2.2.** *Let $P$ be a principal* $\mathrm{Spin}(n)$*-bundle over $M$ with connection $A$, and let $\mathbb{T}$ be a trivialization of $\mathbb{CS}_P$. A <u>string connection</u> is a connection $\blacktriangledown$ on $\mathbb{T}$ compatible with $\nabla_A$. A <u>geometric string structure</u> on $(P, A)$ is a trivialization with string connection.*

We suppress the details behind this definition until Section 3.2. We only remark that a string connection $\blacktriangledown$ on a trivialization $\mathbb{T}$ of $\mathbb{CS}_P$ defines in particular a closed 3-form $K_{\blacktriangledown} \in \Omega^3_{\mathrm{cl}}(P)$ which represents the string structure $\xi_{\mathbb{T}} \in \mathrm{H}^3(P,\mathbb{Z})$ that is determined by $\mathbb{T}$ under the bijection of Theorem 1.1.4.

In the remainder of this subsection we present an argument why our Definition 1.2.2 of a string connection is appropriate. For this purpose we shall compare it with the original Definition 5.3.4 in [ST04]. There, a geometric string structure on $P$ is characterized by its impact on the values of a classical Chern-Simons field theory: it gets "trivialized".

According to a general classification result of Dijkgraaf and Witten, Chern-Simons theories with gauge group a compact Lie group $G$ are classified by $\mathrm{H}^4(BG,\mathbb{Z})$. In the present case of $G = \mathrm{Spin}(n)$ our class is

$$\tfrac{1}{2}p_1 \in \mathrm{H}^4(B\mathrm{Spin}(n),\mathbb{Z}).$$

Thus, we are given a canonical Chern-Simons theory $Z$. For the purposes of this section we want to restrict our attention to one aspect of the Chern-Simons theory $Z$: it assigns a number

$$Z(X,P,A) \in \mathrm{U}(1)$$

to each closed oriented 3-manifold $X$ that is equipped with a principal $\mathrm{Spin}(n)$-bundle $P$ with connection $A$.

Following Stolz and Teichner, we "pull back" $Z$ using a fixed $\mathrm{Spin}(n)$-bundle $P$ with connection $A$ over some smooth manifold $M$. The result is a collection of smooth maps

$$Z_{P,A}(X) : C^\infty(X,M) \longrightarrow \mathrm{U}(1) : f \longmapsto Z(X, f^*P, f^*A),$$

one for each three-dimensional closed oriented manifold $X$.

We say that a *trivialization* of $Z_{P,A}$ is, for each $X$, a lift of $Z_{P,A}(X)$ through a smooth map

$$T(X) : C^\infty(X,M) \longrightarrow \mathbb{R}$$

and the exponential map $\exp(2\pi\mathrm{i}-) : \mathbb{R} \longrightarrow \mathrm{U}(1)$.

**Theorem 1.2.3.** *Let $P$ be a principal $G$-bundle over $M$ with connection $A$. Then, there exists a canonical map*

$$\left\{ \begin{array}{c} \textit{Geometric string} \\ \textit{structures on } (P,A) \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \textit{Trivializations of the} \\ \textit{Chern-Simons theory } Z_{P,A} \end{array} \right\}.$$

We prove this theorem in Section 3.4 using that the smooth maps $Z_{P,A}(X)$ can be identified with the holonomy of the canonical connection $\nabla_A$ on the Chern-Simons 2-gerbe $\mathbb{CS}_P$.

Theorem 1.2.3 shows that our notion of a geometric string structure satisfies one requirement of Definition 5.3.4 in [ST04]. The other requirements concern the values of the Chern-Simons theory $Z_{P,A}$ on closed oriented manifolds $X^d$ of dimensions $d = 0, 1, 2$ and $d = 4$. We discuss these cases in more detail in Section 3.4. For $d = 2, 4$ we obtain further coincidence between the two definitions. Checking the cases $d = 0, 1$ has to be deferred to future work.

## 1.3 Results on String Connections

We have argued that trivializations of the Chern-Simons 2-gerbe $\mathbb{CS}_P$ with compatible connection are an appropriate reformulation of geometric string structures on a pair $(P, A)$ of a principal $\mathrm{Spin}(n)$-bundle $P$ over $M$ and a connection $A$ on $P$. Using this reformulation, we are able to prove the following results on string connections.

Two results are direct consequences of the general theory of bundle 2-gerbes with connection. The first is an analog of Corollary 1.1.5.

**Corollary 1.3.1.** *Geometric string structures on a principal* $\mathrm{Spin}(n)$*-bundle* $P$ *over* $M$ *with connection* $A$ *form a 2-groupoid* $\mathcal{T}riv(\mathbb{CS}_P, \nabla_A)$.

The second result extends the classification Theorem 1.1.2 (b) from string structures to geometric string structures.

**Corollary 1.3.2.** *Let* $P$ *be a principal* $\mathrm{Spin}(n)$*-bundle over* $M$ *with connection* $A$*, and suppose* $\frac{1}{2}p_1(P) = 0$*. Then,* $(P, A)$ *admits a geometric string structure. The set of isomorphism classes of geometric string structures is a torsor over the differential cohomology group* $\hat{\mathrm{H}}^3(M, \mathbb{Z})$.

Here we understand differential cohomology in the universal sense of Simons and Sullivan [SS07]. In the context of bundle gerbes and bundle 2-gerbes, Deligne cohomology is often an appropriate realization. We recall that one feature of differential cohomology is a commutative diagram

$$\begin{tikzcd} \hat{\mathrm{H}}^k(M, \mathbb{Z}) \arrow[r, "\mathrm{pr}"] \arrow[d, "\Omega"'] & \mathrm{H}^k(M, \mathbb{Z}) \arrow[d, "\iota^*"] \\ \Omega^k_{\mathrm{cl}}(M) \arrow[r] & \mathrm{H}^k(M, \mathbb{R}). \end{tikzcd}$$

Acting with a class $\kappa \in \hat{\mathrm{H}}^3(M,\mathbb{Z})$ on a geometric string structure covers the old action of $\mathrm{pr}(\kappa) \in \mathrm{H}^3(M,\mathbb{Z})$ on the underlying string structure.

The next result is an interesting relation between string structures and 3-forms on $M$. Such a relation has been discovered by Redden [Red06] in the presence of a Riemannian metric on $M$.

**Theorem 1.3.3.** *Let $\pi : P \longrightarrow M$ be a principal* $\mathrm{Spin}(n)$*-bundle over $M$ with a connection $A$. Let $(\mathbb{T}, \blacktriangledown)$ be a geometric string structure on $(P, A)$. Then, there exists a unique 3-form $H_\blacktriangledown \in \Omega^3(M)$ such that*

$$\pi^* H_\blacktriangledown = K_\blacktriangledown + TP(A),$$

*where $K_\blacktriangledown$ is the canonical 3-form which represents the string structure $\xi_\mathbb{T}$ associated to $\mathbb{T}$, and $TP(A)$ is the Chern-Simons 3-form associated to the connection $A$. Moreover, $H_\blacktriangledown$ has the following properties:*

1. *Its derivative is one half of the Pontryagin 4-form of $A$.*
2. *It depends only on the isomorphism class of $(\mathbb{T}, \blacktriangledown)$.*
3. *For $\kappa \in \hat{\mathrm{H}}^3(M,\mathbb{Z})$ we have*

   $$H_{\blacktriangledown.\kappa} = H_\blacktriangledown + \Omega(\kappa).$$

   *under the action of Corollary 1.3.2.*

This theorem follows from Lemmata 3.2.4 and 3.2.5. It remains an interesting open question, if a string connection $\blacktriangledown_g$ can be chosen depending on a Riemannian metric $g$ on $M$ such that $H_{\blacktriangledown_g}$ coincides with Redden's 3-form.

Our last result is

**Theorem 1.3.4.** *For $P$ a principal* $\mathrm{Spin}(n)$*-bundle over $M$, let $\mathbb{T}$ be a trivialization of $\mathbb{CS}_P$. For any choice of a connection $A$ on $P$, there exists a string connection $\blacktriangledown$ on $\mathbb{T}$. The set of string connections on $\mathbb{T}$ is an affine space.*

This result verifies a conjecture of Stolz and Teichner (see [ST04], Theorem 5.3.5). For the proof of Theorem 1.3.4 we show in full generality that every trivialization of every bundle 2-gerbe with connection admits a connection (Proposition 3.3.1), and that the set of such connections is an affine space (Proposition 3.3.4).

# 2 Bundle 2-Gerbes and their Trivializations

We provide the definitions of bundle 2-gerbes and their trivializations, describe their relation to string structures and prove the results outlined in Section 1.1.

## 2.1 The Chern-Simons Bundle 2-Gerbe

We give the details for the proof of Theorem 1.1.3: we describe the construction of the Chern-Simons 2-gerbe $\mathbb{CS}_P$ associated to a principal $\mathrm{Spin}(n)$-bundle $P$.

First we recall the definition of a bundle 2-gerbe, which is based very much on the notion a bundle (1-)gerbes introduced by Murray [Mur96]. All bundle gerbes in this article will be principal $\mathrm{U}(1)$-bundle gerbes. Since bundle gerbes become more and more common, I just want to recall a few facts.

1. For $M$ a smooth manifold, bundle gerbes over $M$ form a strictly monoidal 2-groupoid $\mathcal{G}rb(M)$. Its 1-morphisms will be called *isomorphisms*, and its 2-morphisms will be called *transformations*.

2. Denoting by $\mathrm{h}_0\mathcal{G}rb(M)$ the group of isomorphism classes of bundle gerbes over $M$, there exists a canonical group isomorphism

   $$\mathrm{DD} : \mathrm{h}_0\mathcal{G}rb(M) \longrightarrow \mathrm{H}^3(M,\mathbb{Z}).$$

   For $\mathcal{G}$ a bundle gerbe, $\mathrm{DD}(\mathcal{G})$ is called the *Dixmier-Douady class* of $\mathcal{G}$.

3. For $f : M \longrightarrow N$ a smooth map, there is a pullback 2-functor

   $$f^* : \mathcal{G}rb(N) \longrightarrow \mathcal{G}rb(M),$$

   and whenever smooth maps are composable, the associated 2-functors compose strictly.

For convenience, I have provided some basic definitions in Section 4.1. Apart from these, the reader is referred to [SW, Mur07] for introductions, and to [Ste00, Wal07b] for detailed treatments.

Let us fix some notation. We say that a *covering* of a smooth manifold $M$ is a surjective submersion $\pi : Y \longrightarrow M$. We denote the $k$-fold fibre product of $Y$ with itself by $Y^{[k]}$. This is again a smooth manifold, and for integers $k > p$, all the projections

$$\pi_{i_1,...,i_p} : Y^{[k]} \longrightarrow Y^{[p]} : (y_1, ..., y_k) \longmapsto (y_{i_1}, ..., y_{i_p})$$

are smooth.

**Definition 2.1.1** ([Ste04], Def. 5.3)**.** *A* <u>*bundle 2-gerbe*</u> *over $M$ is a covering $\pi : Y \longrightarrow M$ together with a bundle gerbe $\mathcal{P}$ over $Y^{[2]}$, an isomorphism*

$$\mathcal{M} : \pi_{12}^*\mathcal{P} \otimes \pi_{23}^*\mathcal{P} \longrightarrow \pi_{13}^*\mathcal{P}$$

*of bundle gerbes over $Y^{[3]}$, and a transformation*

$$\begin{tikzcd} \pi_{12}^*\mathcal{P} \otimes \pi_{23}^*\mathcal{P} \otimes \pi_{34}^*\mathcal{P} \arrow[r, "\pi_{123}^*\mathcal{M}\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\pi_{234}^*\mathcal{M}"'] & \pi_{13}^*\mathcal{P} \otimes \pi_{34}^*\mathcal{P} \arrow[d, "\pi_{134}^*\mathcal{M}"] \arrow[dl, Rightarrow, "\mu"'] \\ \pi_{12}^*\mathcal{P} \otimes \pi_{24}^*\mathcal{P} \arrow[r, "\pi_{124}^*\mathcal{M}"'] & \pi_{14}^*\mathcal{P} \end{tikzcd}$$

*over $Y^{[4]}$, which satisfies the pentagon axiom.*

The isomorphism $\mathcal{M}$ is called *product* and the transformation $\mu$ is called *associator*. We have sketched the pentagon axiom in Figure 1 on page 40. Every bundle 2-gerbe $\mathbb{G}$ over $M$ defines a *characteristic class* $\mathrm{CC}(\mathbb{G}) \in \mathrm{H}^4(M, \mathbb{Z})$, analogous to the Dixmier-Douady class of a bundle gerbe; see [Ste04], Prop. 7.2.

Now let $G$ be a Lie group. In fact we are only interested in $G = \mathrm{Spin}(n)$, but the following construction applies in general. For more details on this construction we refer the reader to [CJM+05, Wal08].

Let $\pi : P \longrightarrow M$ be a principal $G$-bundle over $M$. The key idea in the construction of the Chern-Simons 2-gerbe $\mathbb{CS}_P$ is to take the bundle projection $\pi : P \longrightarrow M$ as its covering. Its two-fold fibre product comes with a canonical smooth map $g : P^{[2]} \longrightarrow G$ defined by $p'.g(p,p') = p$, for $p, p' \in P$ two points in the same fibre. Suppose now we have a bundle gerbe $\mathcal{G}$ over $G$ available. Then we put

$$\mathcal{P} := g^*\mathcal{G}$$

as the bundle gerbe of $\mathbb{CS}_P$. It turns out that the remaining structure, namely the product $\mathcal{M}$ and the associator $\mu$, can be understood as additional structure for the bundle gerbe $\mathcal{G}$ over $G$, called a *multiplicative structure*. Bundle gerbes with multiplicative structure are called *multiplicative bundle gerbes*.

We have thus a bundle 2-gerbe $\mathbb{CS}_P(\mathcal{G})$ associated to every principal $G$-bundle $P$ and every multiplicative bundle gerbe $\mathcal{G}$ over $G$. We remark that multiplicative bundle gerbes $\mathcal{G}$ over $G$ are classified up to isomorphism by

$\mathrm{H}^4(BG, \mathbb{Z})$ via a *multiplicative class* $\mathrm{MC}(\mathcal{G}) \in \mathrm{H}^4(BG, \mathbb{Z})$ [CJM+05]. The transgression

$$\mathrm{H}^4(BG, \mathbb{Z}) \longrightarrow \mathrm{H}^3(G, \mathbb{Z})$$

takes the multiplicative class to the Dixmier-Douady class of the underlying bundle gerbe, see [Wal08], Prop 2.11.

The relation between the multiplicative class of a multiplicative bundle gerbe $\mathcal{G}$ over $G$ and the characteristic class of the associated Chern-Simons 2-gerbe $\mathbb{CS}_P(\mathcal{G})$ is

**Lemma 2.1.2** ([Wal08], Thm. 3.13)**.** *Let $P$ be a principal $G$-bundle over $M$, and let $\eta : M \longrightarrow BG$ be a classifying map for $P$. Then,*

$$\mathrm{CC}(\mathbb{CS}_P(\mathcal{G})) = \eta^*\mathrm{MC}(\mathcal{G}).$$

Let us restrict to $G = \mathrm{Spin}(n)$, and a assume a principal $\mathrm{Spin}(n)$-bundle $P$ over $M$ with a classifying map $\eta$. Suppose we have a multiplicative bundle gerbe $\mathcal{G}$ over $G$ such that

$$\eta^*\mathrm{MC}(\mathcal{G}) = \tfrac{1}{2}p_1(P) \in \mathrm{H}^4(M, \mathbb{Z}). \tag{2.1.1}$$

Then, by Lemma 2.1.2, we obtain

$$\mathrm{CC}(\mathbb{CS}_P) = \tfrac{1}{2}p_1(P),$$

which proves Theorem 1.1.3.

In order to find a canonical multiplicative bundle gerbe $\mathcal{G}$ satisfying (2.1.1) we infer the following result of McLaughlin.

**Lemma 2.1.3** ([McL92], Lemma 2.2)**.** *Let $P$ be a principal* $\mathrm{Spin}(n)$*-bundle over $M$, and let $\eta : M \longrightarrow B\mathrm{Spin}(n)$ be a classifying map for $P$. There is a unique $\tau \in \mathrm{H}^4(B\mathrm{Spin}(n), \mathbb{Z})$ whose transgression is the preferred generator of* $\mathrm{H}^3(\mathrm{Spin}(n), \mathbb{Z})$*. Moreover,*

$$\eta^*\tau = \tfrac{1}{2}p_1(P).$$

The lemma tells us that the multiplicative bundle gerbe $\mathcal{G}$ we want to find has multiplicative class equal to $\tau$. In turn, this means that the underlying bundle gerbe $\mathcal{G}$ has a Dixmier-Douady class equal to the preferred generator of $\mathrm{H}^3(\mathrm{Spin}(n), \mathbb{Z})$. This bundle gerbe is well-known and called the *basic gerbe* over $\mathrm{Spin}(n)$. It enjoys an explicit, Lie-theoretical canonical construction in the category of smooth, finite-dimensional manifolds, carried out by Gawędzki-Reis [GR02] and Meinrenken [Mei02].

The multiplicative structure on the basic gerbe $\mathcal{G}$ can be constructed in the same manner, although this construction has not appeared in the literature. Anyway, by the above-mentioned classification result for multiplicative bundle gerbes, the multiplicative structure on the basic bundle gerbe is unique at least up to isomorphism. Picking one of these, the construction of the Chern-Simons 2-gerbe $\mathbb{CS}_P$ is completed.

## 2.2 Trivializations and String Structures

Next is the proof of Theorem 1.1.4: we explain the relation between trivializations of the Chern-Simons 2-gerbe $\mathbb{CS}_P$ associated to a principal $\mathrm{Spin}(n)$-bundle $P$ over $M$ and string structures on $P$.

**Definition 2.2.1** ([Ste04], Def. 11.1)**.** *Let $\mathbb{G} = (Y, \mathcal{P}, \mathcal{M}, \mu)$ be a bundle 2-gerbe over $M$. A* <u>*trivialization*</u> *of $\mathbb{G}$ is a bundle gerbe $\mathcal{S}$ over $Y$, together with an isomorphism*

$$\mathcal{A} : \mathcal{P} \otimes \pi_2^*\mathcal{S} \longrightarrow \pi_1^*\mathcal{S}$$

*of bundle gerbes over $Y^{[2]}$ and a transformation*

$$\begin{tikzcd}
\pi_{12}^*\mathcal{P} \otimes \pi_{23}^*\mathcal{P} \otimes \pi_3^*\mathcal{S} \arrow[r, "\mathrm{id}\otimes\pi_{23}^*\mathcal{A}"] \arrow[d, "\mathcal{M}\otimes\mathrm{id}"'] & \pi_{12}^*\mathcal{P} \otimes \pi_2^*\mathcal{S} \arrow[d, "\pi_{12}^*\mathcal{A}"] \arrow[dl, phantom, "\swarrow\sigma"] \\
\pi_{13}^*\mathcal{P} \otimes \pi_3^*\mathcal{S} \arrow[r, "\pi_{13}^*\mathcal{A}"'] & \pi_1^*\mathcal{S}
\end{tikzcd}$$

*over $Y^{[3]}$ which is compatible with the associator $\mu$.*

The compatibility condition is shown in Figure 2 on page 40; it will only become important in Section 5. The purpose of a trivialization is the following.

**Lemma 2.2.2** ([Ste04], Proposition 11.2)**.** *The characteristic class of a bundle 2-gerbe vanishes if and only if it admits a trivialization.*

Applied to the Chern-Simons 2-gerbe $\mathbb{CS}_P$, Lemma 2.2.2 proves the first part of Theorem 1.1.4: $P$ admits a string structure if and only if $\mathbb{CS}_P$ admits a trivialization. The second part follows from Propositions 2.2.3 and 2.2.6 below. The first proposition explains what a trivialization has to do with a string structure.

**Proposition 2.2.3.** *Let $P$ be a principal* $\mathrm{Spin}(n)$*-bundle over $M$, and let $\mathbb{T} = (\mathcal{S}, \mathcal{A}, \sigma)$ be a trivialization of the Chern-Simons 2-gerbe $\mathbb{CS}_P$. Then,*

$$\xi := \mathrm{DD}(\mathcal{S}) \in \mathrm{H}^3(P, \mathbb{Z})$$

*is a string structure on $P$ in the sense of Definition 1.1.1.*

Proof. We have to show that the pullback of $\mathrm{DD}(\mathcal{S})$ along an inclusion $\iota_p : \mathrm{Spin}(n) \longrightarrow P$ is the generator of $\mathrm{H}^3(\mathrm{Spin}(n), \mathbb{Z})$. By construction, this is nothing but the Dixmier-Douady class of the basic gerbe $\mathcal{G}$ which entered the structure of the Chern-Simons 2-gerbe $\mathbb{CS}_P$.

Consider the isomorphism $\mathcal{A} : g^*\mathcal{G} \otimes \pi_2^*\mathcal{S} \longrightarrow \pi_1^*\mathcal{S}$ of bundle gerbes over $P^{[2]}$ which is part of $\mathbb{T}$. For

$$s_p : \mathrm{Spin}(n) \longrightarrow P^{[2]} : g \longmapsto (p.g, p)$$

we observe that $g \circ s_p = \mathrm{id}$, $\pi_1 \circ s_p = \iota_p$, and $\pi_2 \circ s_p$ is a constant map. Thus, the pullback $s_p^*\mathcal{A}$ implies on the Dixmier-Douady classes

$$\mathrm{DD}(\mathcal{G}) = \mathrm{DD}(\iota_p^*\mathcal{S}),$$

since a bundle gerbe over a point has vanishing Dixmier-Douady class. □

For the second proposition we require two lemmata.

**Lemma 2.2.4.** *Trivializations of a fixed bundle 2-gerbe $\mathbb{G}$ form a 2-groupoid denoted $\mathcal{T}riv(\mathbb{G})$.*

This also implies Corollary 1.1.5. For the moment it is enough to note that a 1-morphism between trivializations $(\mathcal{S}_1, \mathcal{A}_1, \sigma_1)$ and $(\mathcal{S}_2, \mathcal{A}_2, \sigma_2)$ involves an isomorphism $\mathcal{B} : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ between the two bundle gerbes. We defer a complete treatment of the 2-groupoid $\mathcal{T}riv(\mathbb{G})$ to Section 5.1.

The next lemma equips the 2-groupoid $\mathcal{T}riv(\mathbb{G})$ with an additional structure.

**Lemma 2.2.5.** *The 2-groupoid $\mathcal{T}riv(\mathbb{G})$ has the structure of a module over the monoidal 2-groupoid $\mathcal{G}rb(M)$ of bundle gerbes over $M$. Moreover:*

*(i) If $\mathbb{T} = (\mathcal{S}, \mathcal{A}, \sigma)$ is a trivialization of $\mathbb{G}$ and $\mathcal{K}$ is a bundle gerbe over $M$, the new trivialization $\mathbb{T}.\mathcal{K}$ has the bundle gerbe $\mathcal{S} \otimes \pi^*\mathcal{K}$.*

*(ii) On isomorphism classes a free and transitive action of the group $\mathrm{h}_0\mathcal{G}rb(M)$ on the set $\mathrm{h}_0\mathcal{T}riv(\mathbb{G})$ is induced.*

The definition of this action and the proof of its properties is an exercise in bundle gerbe theory and deferred to Section 5.2.

Now we can complete the proof of Theorem 1.1.4.

**Proposition 2.2.6.** *Let $P$ be a principal* $\mathrm{Spin}(n)$*-bundle over $M$. Then, the assignment*

$$\left\{ \begin{array}{c} \textit{Isomorphism classes of} \\ \textit{trivializations of } \mathbb{CS}_P \end{array} \right\} \longrightarrow \left\{ \textit{ String structures on } P \right\}$$
$$(\mathcal{S}, \mathcal{A}, \sigma) \longmapsto \mathrm{DD}(\mathcal{S})$$

*defines a bijection.*

Proof. After what we have said about the 1-morphisms between trivializations, the assignment $(\mathcal{S}, \mathcal{A}, \sigma) \longmapsto \mathrm{DD}(\mathcal{S})$ does not depend on the choice of a representative. With Proposition 2.2.3, it is hence well-defined. It is a bijection because it is an equivariant map between $\mathrm{H}^3(M, \mathbb{Z})$-torsors: domain and codomain are torsors due to Lemma 2.2.5 (ii) and Theorem 1.1.2 (b). The equivariance follows from Lemma 2.2.5 (i). $\square$

**Remark 2.2.7.** One can regard a trivialization of a bundle 2-gerbe $\mathbb{G}$ as a "twisted gerbe", analogous to the notion of a *twisted line bundle*, which is nothing but a trivialization of a gerbe (see [Wal07b], Sec 3). This point of view underlies a project of Sati, Schreiber, Škoda and Stevenson [SSSS]. Their approach also allows to consider "higher rank" twisted gerbes, which exist even when $\mathrm{CC}(\mathbb{G})$ is non-trivial but torsion.

# 3 Connections on Bundle 2-Gerbes

We provide definitions of connections on bundle 2-gerbes and on their trivializations, introduce our notion of a string connection and outline the proofs of our results on string connections.

## 3.1 Connections on Chern-Simons 2-Gerbes

We prove Theorem 1.2.1: for $\mathbb{CS}_P$ the Chern-Simons 2-gerbe associated to a principal $\mathrm{Spin}(n)$-bundle $P$, we construct a canonical connection $\nabla_A$ on $\mathbb{CS}_P$ associated to connection $A$ on $P$.

Again, I only want to recall a few facts on connections on bundle gerbes. Basic definitions are provided in Section 4.2, and further available in [Ste00,

Wal07b]. An important feature of a connection $\nabla$ on a bundle gerbe $\mathcal{G}$ is its curvature, which is a closed 3-form $\mathrm{curv}(\nabla) \in \Omega^3(M)$ and represents the Dixmier-Douady class $\mathrm{DD}(\mathcal{G})$ in real cohomology. Bundle gerbes with connection form another 2-groupoid $\mathcal{G}rb^{\nabla}(M)$, which has a forgetful 2-functor

$$\mathcal{G}rb^{\nabla}(M) \longrightarrow \mathcal{G}rb(M).$$

As already mentioned in Section 1.2 one has to be aware that for an isomorphism to be "connection-preserving" is *structure*, not property, see Definition 4.2.3. Thus, the 1-morphisms of $\mathcal{G}rb^{\nabla}(M)$ are *isomorphisms with compatible connection*. The 2-morphisms are *connection-preserving transformations*.

**Definition 3.1.1.** *Let $\mathbb{G} = (Y, \mathcal{P}, \mathcal{M}, \mu)$ be a bundle 2-gerbe over $M$. A connection on $\mathbb{G}$ is a 3-form $B \in \Omega^3(Y)$, together with a connection $\nabla$ on $\mathcal{P}$ of curvature*

$$\mathrm{curv}(\nabla) = \pi_2^*B - \pi_1^*B, \tag{3.1.1}$$

*and a compatible connection on the product $\mathcal{M}$, such that the associator $\mu$ is connection-preserving.*

Analogously to bundle gerbes with connection, every connection on a bundle 2-gerbe has a curvature. It is the unique 4-form $F \in \Omega^4(M)$ such that $\pi^*F = \mathrm{d}B$. It is closed and its cohomology class coincides with the image of $\mathrm{CC}(\mathbb{G})$ in real cohomology ([Ste04], Prop 8.2).

We recall from Section 2.1 that the Chern-Simons 2-gerbe $\mathbb{CS}_P(\mathcal{G})$ is defined from two parameters: the principal $G$-bundle $P$ and a canonical multiplicative bundle gerbe $\mathcal{G}$ over $G$. It turns out that a connection on $\mathbb{CS}_P(\mathcal{G})$ is determined by three parameters: a connection $A$ on $P$, an invariant bilinear form $\langle -, - \rangle$ on the Lie algebra $\mathfrak{g}$ of $G$, and a certain kind of connection on the multiplicative bundle gerbe $\mathcal{G}$ [Wal08].

For the case of $G = \mathrm{Spin}(n)$ we choose the bilinear form to be the Killing form on $\mathfrak{spin}(n)$, normalized such that the closed 3-form

$$H = \tfrac{1}{6} \langle \theta \wedge [\theta \wedge \theta] \rangle \in \Omega^3(\mathrm{Spin}(n)), \tag{3.1.2}$$

with $\theta$ the left-invariant Maurer-Cartan form, represents the image of the preferred element $1 \in \mathrm{H}^3(\mathrm{Spin}(n), \mathbb{Z})$ in real cohomology.

Let us briefly recall the construction of connections on Chern-Simons 2-gerbes $\mathbb{CS}_P(\mathcal{G})$ as described in [Wal08]. The first ingredient, the 3-form on $P$, is the *Chern-Simons 3-form*

$$TP(A) := \langle A \wedge \mathrm{d}A \rangle + \tfrac{2}{3} \langle A \wedge A \wedge A \rangle \in \Omega^3(P).$$

The reason to choose this particular 3-form becomes clear when one computes the difference between its two pullbacks to $P^{[2]}$, which is

$$\pi_2^* TP(A) - \pi_1^* TP(A) = g^*H + \mathrm{d}\omega,$$

with $H$ the 3-form (3.1.2) and a certain 2-form $\omega \in \Omega^2(P^{[2]})$. According to condition (3.1.1), this difference has to coincide with the curvature of the connection on the bundle gerbe $\mathcal{P} = g^*\mathcal{G}$ of $\mathbb{CS}_P(\mathcal{G})$.

The constructions of Gawędzki-Reis and Meinrenken of the basic bundle gerbe $\mathcal{G}$ also include the construction of a canonical connection on $\mathcal{G}$ of curvature $H$. So, the pullback gerbe $\mathcal{P} = g^*\mathcal{G}$ carries a connection of curvature $g^*H$. This connection can further be modified using the 2-form $\omega$ (see Lemma 3.3.5), such that the resulting connection $\nabla_\omega$ on $\mathcal{P}$ has the desired curvature $g^*H + \mathrm{d}\omega$. Thus, the connection $\nabla_\omega$ on $\mathcal{P}$ qualifies as the second ingredient of the connection on $\mathbb{CS}_P$.

It is discussed in detail in [Wal08] how one can continue this construction of the connection $\nabla_A$. The only further structure one needs is a connection on the multiplicative structure on $\mathcal{G}$. As for the multiplicative structure itself, no explicit construction of this connection available in the literature at this point. A general classification result ([Wal08], Cor. 2.6) assures that the connection exists and is unique up to isomorphism. We shall continue by picking one of these, so that we finally obtain a connection $\nabla_A$ on $\mathbb{CS}_P$.

**Lemma 3.1.2** ([Wal08], Thm. 3.13)**.** *The curvature of the connection $\nabla_A$ on the Chern-Simons 2-gerbe $\mathbb{CS}_P$ is the 4-form*

$$\langle \Omega_A \wedge \Omega_A \rangle \in \Omega^4(M),$$

*where $\Omega_A$ is the curvature of $A$.*

Since the curvature of any bundle 2-gerbe represents its characteristic class, which is here $\frac{1}{2}p_1(P)$, it follows that the curvature of $\nabla_A$ is one half of the Pontryagin 4-form of $P$. (The factor $\frac{1}{2}$ does not appear in the formula for the curvature, because we have normalized the bilinear form $\langle -, - \rangle$ with respect to $\mathrm{Spin}(n)$ and not with respect to $\mathrm{SO}(n)$.)

## 3.2 Connections and Geometric String Structures

We give the details of our new notion of a string connection as a compatible connection on a trivialization of the Chern-Simons 2-gerbe $\mathbb{CS}_P$. Then we prove Theorem 1.3.3, the relation between geometric string structures and 3-forms.

**Definition 3.2.1.** *Let $\mathbb{G}$ be a bundle 2-gerbe over $M$ with connection, and let $\mathbb{T} = (\mathcal{S}, \mathcal{A}, \sigma)$ be a trivialization of $\mathbb{G}$. A compatible connection $\blacktriangledown$ on $\mathbb{T}$ is a connection on the bundle gerbe $\mathcal{S}$ and a compatible connection on the isomorphism $\mathcal{A}$, such that the transformation $\sigma$ is connection-preserving.*

The 3-form $K_{\blacktriangledown}$ we have mentioned in Section 1.2 is the curvature of the connection on the bundle gerbe $\mathcal{S}$. Under the bijection of Proposition 2.2.6, it thus represents the string structure $\xi_{\mathbb{T}} = \mathrm{DD}(\mathcal{S})$ in real cohomology.

First we generalize Lemma 2.2.2 to the setup with connection.

**Lemma 3.2.2.** *Let $\mathbb{G}$ be a bundle 2-gerbe with connection. Then,* $\mathrm{CC}(\mathbb{G}) = 0$ *if and only if $\mathbb{G}$ admits a trivialization with compatible connection.*

It is clear that if a trivialization with compatible connection is given, then $\mathrm{CC}(\mathbb{G}) = 0$ by Lemma 2.2.2. We prove the converse in Section 5.3 using the fact that bundle 2-gerbes are classified by degree four differential cohomology.

Concerning the algebraic structure of trivializations with connection, we have

**Lemma 3.2.3.** *Let $\mathbb{G}$ be a bundle 2-gerbe with connection $\nabla$ and* $\mathrm{CC}(\mathbb{G}) = 0$.

*(i) The trivializations of $\mathbb{G}$ with compatible connection form a 2-groupoid $\mathcal{Triv}(\mathbb{G}, \nabla)$.*

*(ii) The 2-groupoid $\mathcal{Triv}(\mathbb{G}, \nabla)$ is a module for the monoidal 2-groupoid $\mathcal{Grb}^{\nabla}(M)$ of bundle gerbes with connection over $M$.*

*(iii) On isomorphism classes, a free and transitive action of the group $\mathrm{h}_0\mathcal{Grb}^{\nabla}(M)$ on the set $\mathrm{h}_0\mathcal{Triv}(\mathbb{G}, \nabla)$ is induced.*

This lemma is a straightforward generalization of our previous results on trivializations without connection, see Remarks 5.1.1 and 5.2.1.

We recall from [MS00], Thm. 4.1, that bundle gerbes with connection are classified by degree three differential cohomology,

$$\mathrm{h}_0\mathcal{Grb}^{\nabla}(M) \cong \hat{\mathrm{H}}^3(M, \mathbb{Z}).$$

Under this identification, the two maps

$$\mathrm{pr} : \hat{\mathrm{H}}^3(M, \mathbb{Z}) \longrightarrow \mathrm{H}^3(M, \mathbb{Z}) \quad \text{and} \quad \Omega : \hat{\mathrm{H}}^3(M, \mathbb{Z}) \longrightarrow \Omega^3_{\mathrm{cl}}(M)$$

are given by the Dixmier-Douady class of the bundle gerbe, and the curvature of its connection, respectively. Together with Lemma 3.2.3, this implies Corollaries 1.3.1 and 1.3.2.

Now we are heading towards the proof of Theorem 1.3.3, the relation between geometric string structures and 3-forms.

**Lemma 3.2.4.** *Let $\mathbb{G}$ be a bundle 2-gerbe with covering $\pi : Y \twoheadrightarrow M$ and a connection with 3-form $B \in \Omega^3(Y)$. Let $\mathbb{T} = (\mathcal{S}, \mathcal{A}, \sigma)$ be a trivialization of $\mathbb{G}$ with compatible connection $\blacktriangledown$, the connection on the bundle gerbe $\mathcal{S}$ denoted by $\nabla$. Then, there exists a unique 3-form $H_\blacktriangledown$ on $M$ such that*

$$\pi^* H_\blacktriangledown = \mathrm{curv}(\nabla) + B.$$

Proof. We can prove this right away. For the 3-form

$$C := \mathrm{curv}(\nabla) + B \in \Omega^3(Y)$$

we compute

$$\begin{aligned} \pi_2^* C - \pi_1^* C &= (\pi_2^* \mathrm{curv}(\nabla) - \pi_1^* \mathrm{curv}(\nabla)) + (\pi_2^* B - \pi_1^* B) \\ &= \mathrm{curv}(\mathcal{P}) - \mathrm{curv}(\mathcal{P}) \\ &= 0. \end{aligned}$$

Here we have used condition (3.1.1) and that the isomorphism

$$\mathcal{A} : \mathcal{P} \otimes \pi_2^* \mathcal{S} \longrightarrow \pi_1^* \mathcal{S}$$

is equipped with a compatible connection (which implies the equality of the curvatures of the target and the source bundle gerbes). The computation means (see the exact sequence (4.2.2)), that $C$ is the pullback of a unique 3-form $H_\blacktriangledown$ along $\pi : Y \twoheadrightarrow M$. $\square$

The 3-form $H_\blacktriangledown$ depends only on the isomorphism class of $\mathbb{T}$, since the curvature of $\mathcal{S}$ does so. Thus, the proof of Theorem 1.3.3 is finished with the following calculation, which follows directly from the definitions.

**Lemma 3.2.5.** *Let $\mathbb{G}$ be a bundle 2-gerbe with connection, and let $\mathbb{T}$ be a trivialization with connection $\blacktriangledown$. The 3-form $H_\blacktriangledown$ has the following properties:*

1. $\mathrm{d}H_\blacktriangledown = \mathrm{curv}(\mathbb{G})$.
2. $H_{\blacktriangledown.\mathcal{K}} = \mathrm{curv}(\mathcal{K}) + H_\blacktriangledown$

*where $\mathcal{K}$ is a bundle gerbe with connection over $M$, and $\blacktriangledown.\mathcal{K}$ denotes the action from Lemma 3.2.3 (ii).*

## 3.3 Existence and Classification of compatible Connections on Trivializations

Concerning our results on string connections, we are left with the proof of Theorem 1.3.4. In fact a more general statement is true for trivializations of any bundle 2-gerbe. The first part is

**Proposition 3.3.1.** *Suppose $\mathbb{T}$ is a trivialization of a bundle gerbe $\mathbb{G}$, and suppose $\nabla$ is a connection on $\mathbb{G}$. Then, there exists a connection $\blacktriangledown$ on $\mathbb{T}$ compatible with $\nabla$.*

For $\mathbb{G} = \mathbb{CS}_P$, Proposition 3.3.1 implies the first part of Theorem 1.3.4. Its proof requires two lemmata.

**Lemma 3.3.2** ([Mur96], Sec. 6)**.** *Let $\mathcal{K}$ be a bundle gerbe over $M$. Then, $\mathcal{K}$ admits a connection.*

**Lemma 3.3.3.** *Let $\mathbb{G}$ be a bundle 2-gerbe, let $\mathbb{T}_1$ and $\mathbb{T}_2$ be trivializations and let $\mathbb{B} : \mathbb{T}_1 \longrightarrow \mathbb{T}_2$ be a 1-morphism. Let $\nabla$ be a connection on $\mathbb{G}$, and let $\blacktriangledown$ be a connection on $\mathbb{T}_2$ compatible with $\nabla$. Then, there exists a connection on $\mathbb{T}_1$ compatible with $\nabla$ such that $\mathbb{B}$ becomes a 1-morphism in $\mathcal{T}riv(\mathbb{G}, \nabla)$.*

The proof of this lemma is carried out in Section 5.4. Now we give the proof of the above proposition.

Proof of Proposition 3.3.1. Since $\mathbb{G}$ has by assumption the trivialization $\mathbb{T}$, we have $\mathrm{CC}(\mathbb{G}) = 0$ by Lemma 2.2.2. Hence, by Lemma 3.2.2, there exists a trivialization $\mathbb{T}'$ of $\mathbb{G}$ with connection $\blacktriangledown'$ compatible with $\nabla$. Of course $\mathbb{T}'$ is not necessarily equal or isomorphic to the given trivialization $\mathbb{T}'$, but by Lemma 2.2.5 (ii) the isomorphism classes of trivializations of $\mathbb{G}$ form a torsor over $\mathrm{h}_0\mathcal{G}rb(M)$. Thus, there exists a bundle gerbe $\mathcal{K}$ over $M$ and a 1-morphism $\mathbb{B} : \mathbb{T} \longrightarrow \mathcal{K}.\mathbb{T}'$. According to Lemma 3.3.2 every bundle gerbe admits a connection; so we may choose one on $\mathcal{K}$. Now we use the action of Lemma 3.2.3 (ii) according to which $\mathcal{K}.\mathbb{T}'$ also has a compatible connection. Finally, by Lemma 3.3.3, the 1-morphism $\mathbb{B}$ induces a compatible connection on $\mathbb{T}$. $\square$

Now we want to describe the space of compatible connections on a fixed trivialization $\mathbb{T} = (\mathcal{S}, \mathcal{A}, \sigma)$ of a bundle 2-gerbe $\mathbb{G}$ with a covering $\pi : Y \longrightarrow M$. In order to make the following statements, we have to infer that part of the structure of the bundle gerbe $\mathcal{S}$ over $Y$ is another covering

$\omega : W \longrightarrow Y$ (see Definition 4.1.1). The following vector space $V_{\mathbb{T}}$ associated to the trivialization $\mathbb{T}$ will be relevant. It is the quotient

$$V_{\mathbb{T}} := \left(\Omega^2(M) \oplus \Omega^1(Y) \oplus \Omega^1(W)\right) \,/\, U,$$

where the linear subspace $U$ we divide out is given by

$$U := \left\{ (\mathrm{d}\chi, \pi^*\chi + \nu, \omega^*\nu) \mid \chi \in \Omega^1(M),\, \nu \in \Omega^1(Y) \right\}.$$

Now, the second part of Theorem 1.3.4 is implied by

**Proposition 3.3.4.** *Suppose $\mathbb{T} = (\mathcal{S}, \mathcal{A}, \sigma)$ is a trivialization of $\mathbb{G}$, and suppose $\nabla$ is a connection on $\mathbb{G}$. Then, the set of compatible connections on $\mathbb{T}$ is an affine space over $V_{\mathbb{T}}$.*

The proof of Proposition 3.3.4 is quite technical. Most of the work is deferred to Sections 5.5 and 5.6. At this place we want at least give a hint why the vector space $V_{\mathbb{T}}$ appears. We require the following two lemmata. The first describes how to act on the set of connections of a fixed bundle gerbe.

**Lemma 3.3.5.** *The set of connections on a bundle gerbe $\mathcal{S}$ over $Y$ is an affine space over the real vector space*

$$V_{\mathcal{S}} := \left(\Omega^2(Y) \oplus \Omega^1(W)\right) / \left(\mathrm{d} \oplus \omega^*\right) \Omega^1(Y),$$

*where $\omega : W \longrightarrow Y$ is the covering of $\mathcal{S}$.*

We prove this lemma in Section 4.2 based on results of Murray. The second lemma describes how to act on the set of connections on an isomorphism. Here we have to infer that also an isomorphism comes with its own covering.

**Lemma 3.3.6.** *Let $\mathcal{A} : \mathcal{G} \longrightarrow \mathcal{H}$ be an isomorphism between bundle gerbes with covering $Z$. Then, the set of connections on $\mathcal{A}$ is an affine space over $\Omega^1(Z)$.*

Notice that this is a statement on the set of connections that are *not* necessarily compatible with connections on the bundle gerbes $\mathcal{G}$ and $\mathcal{H}$. Its (obvious) proof can also be found in Section 4.2.

Proof of Proposition 3.3.4. Let us describe the action of the vector space $V_{\mathbb{T}}$ on the set of (not necessarily compatible) connections on $\mathbb{T} = (\mathcal{S}, \mathcal{A}, \sigma)$. For $(\psi, \rho, \varphi) \in V_{\mathbb{T}}$, consider the pair $(\eta, \varphi) \in V_{\mathcal{S}}$ with $\eta := \mathrm{d}\rho - \pi^*\psi \in \Omega^2(Y)$. It operates on the connection on the bundle gerbe $\mathcal{S}$ according to Lemma

3.3.5. For $Z$ the covering space of the 1-isomorphism $\mathcal{A} : \mathcal{P} \otimes \pi_2^*\mathcal{S} \longrightarrow \pi_1^*\mathcal{S}$ of bundle gerbes over $Y^{[2]}$, $Z$ has a projection $p : Z \longrightarrow W \times_M W$. Consider

$$\epsilon := p^*(\delta(\varphi - \omega^*\rho) \in \Omega^1(Z),$$

where $\delta$ is the linear map

$$\delta := \omega_2^* - \omega_1^* : \Omega^1(W) \longrightarrow \Omega^1(W \times_M W),$$

(cf. Lemma 4.2.2). The 1-form $\epsilon$ operates on the 1-isomorphism $\mathcal{A}$ according to Lemma 3.3.6.

It is straightforward to check that this action is well-defined under dividing out the subvectorspace $U$: suppose we have 1-forms $\chi \in \Omega^1(M)$ and $\nu \in \Omega^1(Y)$ and act by the triple consisting of $\psi := \mathrm{d}\chi$, $\rho := \pi^*\chi + \nu$ and $\varphi := \omega^*\nu$. It follows that $\eta = \mathrm{d}\nu$ so that $(\eta, \varphi) \in V_{\mathcal{S}}$ acts trivially by Lemma 3.3.5. Furthermore, we find $\varphi - \omega^*\rho = \omega^*\pi^*\chi$, so that its alternating sum $\delta$ vanishes (see again Lemma 4.2.2). Hence, $\epsilon = 0$.

The remaining steps are the content of the following lemma, which is to be proven in Sections 5.5 and 5.6. □

**Lemma 3.3.7.** *The action of $V_{\mathbb{T}}$ on connections on $\mathbb{T}$ has the following properties:*

*(a) It takes compatible connections to compatible connections.*

*(b) It is free and transitive on compatible connections.*

## 3.4 Trivializing Chern-Simons Theory

In this section we prove Theorem 1.2.3: we show that every geometric string structure on a principal $\mathrm{Spin}(n)$-bundle $P$ with connection $A$ gives rise to a trivialization of the Chern-Simons theory $Z_{P,A}$. The relation between $Z_{P,A}$ and the Chern-Simons 2-gerbe $\mathbb{CS}_P$ and its canonical connection $\nabla_A$ has been described in detail in [Wal08]. It can be expressed in terms of the holonomy of the connection $\nabla_A$, which is – for each closed oriented three-dimensional manifold $X$ – a smooth map

$$\mathrm{Hol}_{\nabla_A} : C^\infty(X, M) \longrightarrow \mathrm{U}(1).$$

Then, the Chern-Simons theory $Z_{P,A}$ is defined by

$$Z_{P,A}(X)(f) := \mathrm{Hol}_{\nabla_A}(f).$$

In order to see why a geometric string structure $(\mathbb{T}, \blacktriangledown)$ on $(P, A)$ determines a trivialization of $Z_{P,A}$ we recall how the holonomy is defined.

Let $\mathbb{G}$ be a bundle 2-gerbe with connection $\nabla$ over $M$. The pullback of $\mathbb{G}$ along $f : X \longrightarrow M$ is a bundle 2-gerbe over $X$ with $\mathrm{CC}(f^*\mathbb{G}) = 0$, since $X$ is three-dimensional. Hence, by Lemma 3.2.2 there exists a trivialization $\mathbb{T}_f$ with compatible connection $\blacktriangledown_f$. Let $H_{\blacktriangledown_f}$ be the canonical 3-form associated to $(\mathbb{T}_f, \blacktriangledown_f)$ by Lemma 3.2.4. Then,

$$\mathrm{Hol}_{\nabla}(f) := \exp\left(2\pi\mathrm{i}\int_X H_{\blacktriangledown_f}\right).$$

One can show that 3-forms $H_{\blacktriangledown_f}$ and $H_{\blacktriangledown_f'}$ corresponding to different choices of trivializations differ by a closed 3-form with integral periods, so that the latter expression is independent of the choice of the trivialization.

Notice that we have chosen, separately for each $f : X \longrightarrow M$, a trivialization $(\mathbb{T}_f, \blacktriangledown_f)$ of the pullback $f^*\mathbb{G}$. If a "global" trivialization $(\mathbb{T}, \blacktriangledown)$ of $\mathbb{G}$ over $M$ is given, we can chose $\mathbb{T}_f := f^*\mathbb{T}$ and $\blacktriangledown_f := f^*\blacktriangledown$ which gives indeed a trivialization of $f^*\mathbb{G}$ with connection compatible with $f^*\nabla_A$. It is straightforward to check that $H_{\blacktriangledown_f} = f^*H_{\blacktriangledown}$. Now we see that the smooth map

$$T(X) : C^\infty(X, M) \longrightarrow \mathbb{R} : f \longmapsto \int_X f^*H_{\blacktriangledown}$$

lifts $\mathrm{Hol}_{\nabla_A}$, and this proves Theorem 1.2.3.

In the remainder of this section we discuss the cases of $d$-dimensional manifolds $X^d$ for $d = 2$ and $d = 4$. First, for $d = 4$, the Chern-Simons theory assigns to a four-dimensional closed oriented manifold $X^4$ the map

$$Z_{P,A}(X^4) : C^\infty(X^4, M) \longrightarrow \mathbb{Z} : f \longmapsto \left\langle [X^4], f^*(\tfrac{1}{2}p_1(P))\right\rangle,$$

where $[X^4]$ is the fundamental class of $X^4$ and $\langle -, - \rangle$ denotes the pairing between homology and cohomology. The condition of Stolz and Teichner requires this map to be identically zero. This is indeed the case: since we have a string structure on $P$, $\frac{1}{2}p_1(P)$ vanishes by Theorem 1.1.4, and so does the pairing with the fundamental class $[X^4]$.

Consider now a closed oriented surface $X^2$. The Chern-Simons theory $Z_{P,A}$ associates to it a principal U(1)-bundle over $C^\infty(X^2, M)$. According to the general prescription of Freed [Fre02], this principal U(1)-bundle is the *transgression* of the Chern-Simons 2-gerbe $\mathbb{CS}_P$ and its connection $\nabla_A$. At this time there is no general theory available for the transgression of bundle 2-gerbes, but we can at least describe the fibres of the resulting U(1)-bundle.

For a smooth map $f : X^2 \longrightarrow M$, consider the set $\mathrm{h}_0\mathcal{T}riv(f^*\mathbb{CS}_P, f^*\nabla_A)$ of isomorphism classes of trivializations of $f^*\mathbb{CS}_P$ with connection compatible with $f^*\nabla_A$. By Lemma 3.2.3 (iii), this set is a torsor over $\hat{\mathrm{H}}^3(X^2,\mathbb{Z})$. We use the exact sequence

$$0 \longrightarrow \mathrm{H}^2(M,\mathrm{U}(1)) \longrightarrow \hat{\mathrm{H}}^3(M,\mathbb{Z}) \xrightarrow{\Omega} \Omega^3_{\mathrm{cl}}(M)$$

in the differential cohomology of any smooth manifold $M$ [SS07]. Since there are no 3-forms on the surface $X^2$, we have

$$\hat{\mathrm{H}}^3(X^2,\mathbb{Z}) \cong \mathrm{H}^2(X^2,\mathrm{U}(1)) \cong \mathrm{U}(1).$$

We have thus associated to each point $f \in C^\infty(X^2, M)$ a $\mathrm{U}(1)$-torsor – this is the fibre of the principal $\mathrm{U}(1)$-bundle.

It is now clear that a geometric string structure $(\mathbb{T}, \blacktriangledown)$ on $(P, A)$ selects an element in each of these torsors, namely the isomorphism class of the trivialization $(f^*\mathbb{T}, f^*\blacktriangledown)$. This matches again the prescription of Definition 5.3.4 in [ST04].

Summarizing, our definition of a geometric string structure coincides with the prescription of Stolz and Teichner in dimensions $d = 2, 3, 4$.

# 4 Background on Bundle Gerbes

This section introduces some of the basic definitions concerning bundle gerbes and connections on bundle gerbes on the basis of [Wal07b]. There are a few new results on the duality for bundle gerbes and on the spaces of connections on bundle gerbes and isomorphisms.

## 4.1 Bundle Gerbes

Let $M$ be a smooth manifold. We recall that a *covering* is a surjective submersion, and we refer to Section 2.1 for our conventions concerning fibre products and the labelling of projections.

**Definition 4.1.1** ([Mur96], Sec. 3)**.** *A <u>bundle gerbe</u> over $M$ is a covering $\pi : Y \longrightarrow M$ together with a principal* $\mathrm{U}(1)$*-bundle $P$ over $Y^{[2]}$ and a bundle isomorphism*

$$\mu : \pi_{12}^*P \otimes \pi_{23}^*P \longrightarrow \pi_{13}^*P$$

*over $Y^{[3]}$ which is associative over $Y^{[4]}$.*

The notion of an isomorphism between bundle gerbes took some time to develop; most appropriate for our purposes is the following generalization of a "stable isomorphism" [MS00]. We consider two bundle gerbes $\mathcal{G}_1$ and $\mathcal{G}_2$ over $M$, whose structure is denoted in the same way as in Definition 4.1.1 but with indices 1 or 2.

**Definition 4.1.2** ([Wal07b], Def. 2)**.** *An* <u>*isomorphism*</u> $\mathcal{A} : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ *is a covering* $\zeta : Z \longrightarrow Y_1 \times_M Y_2$ *together with a principal* $\mathrm{U}(1)$*-bundle* $Q$ *over* $Z$ *and a bundle isomorphism*

$$\alpha : P_1 \otimes \zeta_2^* Q \longrightarrow \zeta_1^* Q \otimes P_2$$

*over* $Z \times_M Z$*, which satisfies a compatibility condition with* $\mu_1$ *and* $\mu_2$*.*

The canonical example of an isomorphism is the identity isomorphism $\mathrm{id}_{\mathcal{G}}$ of a bundle gerbe $\mathcal{G}$. It has the identity covering $\zeta := \mathrm{id}_{Y^{[2]}}$, the principal bundle $Q := P$ and an isomorphism $\alpha$ defined from the isomorphism $\mu$.

Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are two isomorphisms from $\mathcal{G}_1$ to $\mathcal{G}_2$.

**Definition 4.1.3** ([Wal07b], Def. 3)**.** *A* <u>*transformation*</u> $\beta : \mathcal{A}_1 \Longrightarrow \mathcal{A}_2$ *is a covering*

$$k : V \longrightarrow Z_1 \times_{(Y_1 \times_M Y_2)} Z_2$$

*together with a bundle isomorphism* $\beta_V : Q_1 \longrightarrow Q_2$ *between the pullbacks of the bundles of* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *to* $V$*, which satisfies a compatibility condition with the isomorphisms* $\alpha_1$ *and* $\alpha_2$*.*

Additionally, transformations $(V_1, \beta_{V_1})$ and $(V_2, \beta_{V_2})$ are identified whenever the bundle isomorphisms $\beta_{V_1}$ and $\beta_{V_2}$ agree after being pulled back to the fibre product of $V_1$ and $V_2$.

All the operations which turn bundle gerbes, isomorphisms and transformations into a monoidal 2-groupoid are straightforward to find, and we refer to Section 1 of [Wal07b].

We shall say some words on the duality for the monoidal 2-category of bundle gerbes that we need in Section 5.2. It assigns to every bundle gerbe $\mathcal{G}$ a *dual bundle gerbe* $\mathcal{G}^*$, to every isomorphism $\mathcal{A} : \mathcal{G} \longrightarrow \mathcal{H}$ a *dual isomorphism* $\mathcal{A}^* : \mathcal{H}^* \longrightarrow \mathcal{G}^*$, and to every transformation $\beta : \mathcal{A}_1 \Longrightarrow \mathcal{A}_2$ a *dual transformation* $\beta^* : \mathcal{A}_2^* \Longrightarrow \mathcal{A}_1^*$. The definition of these assignments can be found in [Wal07b], Sec 1.3. Basically, the dual bundle gerbe $\mathcal{G}^*$ has the dual principal $\mathrm{U}(1)$-bundle $P^*$ (i.e. the same set but with $\mathrm{U}(1)$ acting by inverses). The dual isomorphism $\mathcal{A}^*$ has the *same* principal $\mathrm{U}(1)$-bundle $Q$ as before, and the dual transformation also has the same bundle isomorphism as before.

There are canonical isomorphisms $\mathcal{D}_{\mathcal{G}} : \mathcal{G}^* \otimes \mathcal{G} \longrightarrow \mathcal{I}$, with $\mathcal{I}$ the tensor unit of the monoidal 2-groupoid of bundle gerbes. The isomorphism $\mathcal{D}_{\mathcal{G}}$ has the identity covering $\zeta := \mathrm{id}_{Y^{[2]}}$, the dual bundle $Q := P^*$, and its isomorphism is defined from the bundle isomorphism $\mu$, see [Wal07a] Sec. 1.2 for a detailed definition. There is still no treatment available that exhibits $\mathcal{D}_{\mathcal{G}}$ as the unit of the duality on the monoidal 2-groupoid of bundle gerbes, and this article is certainly not the place to give one.

However, in Section 5.2 we need two properties of $\mathcal{D}_{\mathcal{G}}$. The first is the existence of a canonical transformation

$$\begin{tikzcd} \mathcal{G}^* \otimes \mathcal{G} \arrow[rr, "\mathcal{A}^{*-1}\otimes\mathcal{A}"] \arrow[dr, "\mathcal{D}_{\mathcal{G}}"'] & \arrow[d, phantom, "\overset{\varphi_{\mathcal{A}}}{\Longleftarrow}"] & \mathcal{H}^* \otimes \mathcal{H} \arrow[dl, "\mathcal{D}_{\mathcal{H}}"] \\ & \mathcal{I} & \end{tikzcd}$$

associated to every isomorphism $\mathcal{A} : \mathcal{G} \longrightarrow \mathcal{H}$. It can be seen as a "naturality" property of $\mathcal{D}_{\mathcal{G}}$. The second property is the existence of a canonical transformation between the isomorphism

$$\mathcal{G} \xrightarrow{\mathrm{id}\otimes\mathcal{D}_{\mathcal{G}}} \mathcal{G} \otimes \mathcal{G}^* \otimes \mathcal{G} \xrightarrow{\mathcal{D}_{\mathcal{G}^*}\otimes\mathrm{id}} \mathcal{G}$$

and the identity isomorphism $\mathrm{id}_{\mathcal{G}}$. It can be seen as a "zigzag" property for $\mathcal{D}_{\mathcal{G}}$. The construction of these two transformation is straightforward and left as an exercise.

## 4.2 Connections

Let $\mathcal{G} = (Y, P, \mu)$ be a bundle gerbe over $M$.

**Definition 4.2.1** ([Mur96], Sec. 6)**.** *A connection on $\mathcal{G}$ is a 2-form $C \in \Omega^2(Y)$ and a connection $\omega$ on $P$ of curvature* $\mathrm{curv}(\omega) = \pi_2^*C - \pi_1^*C$*, such that $\mu$ is connection-preserving.*

One tool which will be frequently used in the following is

**Lemma 4.2.2** ([Mur96], Sec. 8)**.** *Let $\pi : Y \longrightarrow M$ be a surjective submersion and $p \geq 0$ be an integer. Consider*

$$\delta_\pi := \sum_{i=1}^{k} (-1)^k \pi^*_{1,\dots,i-1,i+1,\dots,k} : \Omega^p(Y^{[k-1]}) \longrightarrow \Omega^p(Y^{[k]}).$$

*Then, the sequence*

$$0 \longrightarrow \Omega^p(M) \xrightarrow{\pi^*=\delta_\pi} \Omega^p(Y) \xrightarrow{\delta_\pi} \Omega^p(Y^{[2]}) \xrightarrow{\delta_\pi} \cdots$$

*is exact.*

For example, we see that for a connection $(C,\omega)$ on a bundle gerbe $\mathcal{G}$,

$$\delta_\pi \mathrm{d}C = \pi_2^*\mathrm{d}C - \pi_1^*\mathrm{d}C = \mathrm{dcurv}(\omega) = 0.$$

This means that $\mathrm{d}C$ is the pullback of a unique 3-form $H \in \Omega^3(M)$. This 3-form is the curvature of the connection $(C,\omega)$. We can now give the

Proof of Lemma 3.3.5 from Section 3.3. We have to show that the set of connections on $\mathcal{G}$ is an affine space over the vector space

$$V_{\mathcal{G}} := \big(\Omega^2(M) \oplus \Omega^1(Y)\big)\,/\,(\mathrm{d} \oplus \pi^*)\,\Omega^1(M).$$

We shall first define the action and then show that it is free and transitive.

1. If $(C,\omega)$ is a connection on $\mathcal{G}$, and $(\eta,\varphi) \in V_{\mathcal{G}}$, we have a new connection $(C',\omega')$ defined by $C' := C + \mathrm{d}\varphi - \pi^*\eta$ and $\omega' := \omega + \delta_\pi\varphi$. Indeed,

   $$\mathrm{curv}(\omega') = \mathrm{curv}(\omega) + \mathrm{d}\delta_\pi\varphi = \delta_\pi(C + \mathrm{d}\varphi) = \delta_\pi(C' + \pi^*\eta) = \delta_\pi C',$$

   which is the first condition. Further, since $\delta_\pi(\delta_\pi\varphi) = 0$, the isomorphism $\mu$ over $Y^{[3]}$ preserves $\omega'$. This shows that $(C',\omega')$ is again a connection on $\mathcal{G}$. It is clear that this action is well-defined on the quotient $V_{\mathcal{G}}$.

2. The action is free: suppose $(\eta,\varphi) \in V_{\mathcal{G}}$ acts trivially. Then, $\delta_\pi\varphi = 0$ so that there exists $\nu \in \Omega^1(M)$ with $\pi^*\nu = \varphi$ by Lemma 4.2.2. Further, $\mathrm{d}\varphi - \pi^*\eta = 0$ which implies $\pi^*(\mathrm{d}\nu - \eta) = 0$. Since $\pi^*$ is injective by Lemma 4.2.2, $\eta = \mathrm{d}\nu$. Thus, $(\eta,\varphi) = 0$ in the quotient space $V_{\mathcal{G}}$.

3. The action is transitive: suppose $(C,\omega)$ and $(C',\omega')$ are connections. Then, there is a 1-form $\psi \in \Omega^1(Y^{[2]})$ such that $\omega' = \omega + \psi$. Since $\mu$ is connection-preserving for both $\omega$ and $\omega'$, we see that $\delta_\pi\psi = 0$. By Lemma 4.2.2, there is a 1-form $\varphi \in \Omega^1(Y)$ such that $\delta_\pi\varphi = \psi$. We compute

   $$\delta_\pi(C' - C - \mathrm{d}\psi) = \mathrm{curv}(\omega') - \mathrm{curv}(\omega) - \delta_\pi\mathrm{d}\psi = \mathrm{d}\psi - \mathrm{d}\delta_\pi\psi = 0,$$

   which means that there exists $\eta \in \Omega^1(M)$ with $-\pi^*\eta = C' - C - \mathrm{d}\psi$. Then, the action of $(\eta,\varphi) \in V_{\mathcal{G}}$ takes $(C,\omega)$ to $(C',\omega')$.

We also remark that the action of an element $(\eta,\varphi) \in V_{\mathcal{G}}$ changes the curvature of $\mathcal{G}$ by $\mathrm{d}\eta$. $\square$

Next is the discussion of connections on isomorphisms.

**Definition 4.2.3.** *Let $\mathcal{A} : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ be an isomorphism between bundle gerbes over $M$.*

1. *A connection on $\mathcal{A}$ is a connection $\kappa$ on its principal* $\mathrm{U}(1)$*-bundle $Q$.*
2. *Suppose $\nabla_1 = (C_1, \omega_1)$ and $\nabla_2 = (C_2, \omega_2)$ are connections on $\mathcal{G}_1$ and $\mathcal{G}_2$. Then, a connection $\kappa$ on $\mathcal{A}$ is called compatible with $\nabla_1$ and $\nabla_2$, if* $\mathrm{curv}(\kappa) = \zeta^*(C_2 - C_1)$ *and $\alpha$ is connection-preserving.*

Now we see immediately the claim of Lemma 3.3.6: the (non-compatible) connections on $\mathcal{A}$ are an affine space over the 1-forms on $Z$.

The next lemma shows how to pullback a bundle gerbe connection along an isomorphism.

**Lemma 4.2.4.** *Suppose $\mathcal{A} : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ is a 1-isomorphism, and $\nabla_2$ is a connection on $\mathcal{G}_2$. Then, there exists a connection $\kappa$ on $\mathcal{A}$ and a connection $\nabla_1$ on $\mathcal{G}_1$, such that $\kappa$ is compatible with $\nabla_1$ and $\nabla_2$.*

Proof. We choose any connection $(C_1, \omega_1)$ on the bundle gerbe $\mathcal{G}_1$, and any connection $\kappa$ on $\mathcal{A}$. We compare the pullback connection $\alpha^*(\zeta_1^*\kappa + \omega_2)$ with the connection $\omega_1 + \zeta_2^*\kappa$ on $P_1 \otimes \zeta_2^* Q$. They differ by a 1-form $\beta \in \Omega^1(Z^{[2]})$. The condition on $\alpha$ implies that $\delta\beta = 0$ over $Z^{[3]}$, so that there exists a 1-form $\gamma \in \Omega^1(Z)$ with $\beta = \delta\gamma$. Operating with $\gamma$ on $\kappa$, we obtain a new connection $\kappa'$ on $\mathcal{A}$ such that $\alpha$ is connection-preserving. Consider now the 2-form

$$B := \mathrm{curv}(\kappa') - C_2 + C_1 \in \Omega^2(Z).$$

One readily computes $\delta B = 0$ over $Z^{[2]}$ so that there exists a 2-form $\eta \in \Omega^2(M)$ such that $B = \pi^*\eta$. Operating with $\eta$ on the connection $(C_1, \omega_1)$ yields a new connection on $\mathcal{G}_1$ for which $\kappa'$ is compatible. $\square$

Finally, we need to pullback connections on isomorphisms along transformations.

**Lemma 4.2.5.** *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be bundle gerbes with connection, $\mathcal{A} : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ and $\mathcal{A}' : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ be isomorphisms, and $\beta : \mathcal{A} \Longrightarrow \mathcal{A}'$ is a transformation. Suppose $\kappa$ is a compatible connection on $\mathcal{A}'$. Then, there exists a unique compatible connection on $\mathcal{A}$ such that $\beta$ is connection-preserving.*

This connection is just the pullback of the connection $\kappa$ on $Q'$ along the bundle isomorphism $\beta_V : Q \longrightarrow Q'$.

# 5 Technical Details

Here we provide the proofs of the remaining lemmata.

## 5.1 Lemma 2.2.4: Trivializations form a 2-Groupoid

We prove Lemma 2.2.4: we define a 2-groupoid $\mathcal{T}riv(\mathbb{G})$ of trivializations of a bundle 2-gerbe $\mathbb{G}$. The bundle 2-gerbe $\mathbb{G}$ may consist of a covering $\pi : Y \longrightarrow M$, a bundle gerbe $\mathcal{P}$ over $Y^{[2]}$, a product $\mathcal{M}$ over $Y^{[3]}$ and an associator $\mu$ over $Y^{[4]}$.

We recall from Definition 2.2.1 that a trivialization $\mathbb{T}$ of $\mathbb{G}$ consists of a bundle gerbe $\mathcal{S}$ over $Y$, an isomorphism $\mathcal{A} : \mathcal{P} \otimes \pi_2^*\mathcal{S} \longrightarrow \pi_1^*\mathcal{S}$ over $Y^{[2]}$, and a transformation $\sigma$. Given trivializations $\mathbb{T} = (\mathcal{S}, \mathcal{A}, \sigma)$ and $\mathbb{T}' = (\mathcal{S}', \mathcal{A}', \sigma')$ of $\mathbb{G}$, a *1-morphism* $\mathbb{B} : \mathbb{T} \longrightarrow \mathbb{T}'$ is an isomorphism $\mathcal{B} : \mathcal{S} \longrightarrow \mathcal{S}'$ between bundle gerbes over $Y$ together with a transformation

$$\begin{tikzcd}
\mathcal{P} \otimes \pi_2^*\mathcal{S} \arrow[r, "\mathcal{A}"] \arrow[d, "\mathrm{id}\otimes\pi_2^*\mathcal{B}"'] & \pi_1^*\mathcal{S} \arrow[d, "\pi_1^*\mathcal{B}"] \arrow[dl, Rightarrow, "\beta"] \\
\mathcal{P} \otimes \pi_2^*\mathcal{S}' \arrow[r, "\mathcal{A}'"'] & \pi_1^*\mathcal{S}'
\end{tikzcd}$$

over $Y^{[2]}$ which is compatible with the transformations $\sigma$ and $\sigma'$ in the sense of the pentagon diagram shown in Figure 3 on page 41.

The identity 1-morphism as well as the composition between 1-morphisms are straightforward to find using the structure of the 2-groupoid of bundle gerbes.

If $\mathbb{B}_1 = (\mathcal{B}_1, \beta_1)$ and $\mathbb{B}_2 = (\mathcal{B}_2, \beta_2)$ are both 1-morphisms between $\mathbb{T}$ and $\mathbb{T}'$, a *2-morphism* is a transformation $\varphi : \mathcal{B}_1 \Longrightarrow \mathcal{B}_2$ which is compatible with the transformations $\beta_1$ and $\beta_2$ in such a way that the diagram

$$\begin{tikzcd}
\pi_1^*\mathcal{B}_1 \circ \mathcal{A} \arrow[r, Rightarrow, "\beta_1"] \arrow[d, Rightarrow, "\pi_1^*\varphi\circ\mathrm{id}"'] & \mathcal{A}' \circ (\mathrm{id} \otimes \pi_2^*\mathcal{B}_1) \arrow[d, Rightarrow, "\mathrm{id}\circ(\mathrm{id}\otimes\pi_2^*\varphi)"] \\
\pi_1^*\mathcal{B}_2 \circ \mathcal{A} \arrow[r, Rightarrow, "\beta_2"'] & \mathcal{A}' \circ (\mathrm{id} \otimes \pi_2^*\mathcal{B}_2)
\end{tikzcd}$$

is commutative. Horizontal and vertical composition of 2-morphisms are the ones of the 2-groupoid of bundle gerbes.

It is clear that every 2-morphism is invertible, since every transformation is invertible. In the same way, every 1-morphism is invertible (up to 2-morphisms), since every isomorphism between bundle gerbes is invertible.

The axioms of the 2-groupoid $\mathcal{T}riv(\mathbb{G})$ can easily be deduced from those of the 2-groupoid of bundle gerbes.

**Remark 5.1.1.** If $\nabla$ is a connection on the bundle 2-gerbe $\mathbb{G}$, it is straightforward to repeat the above definitions in the 2-category of bundle gerbe *with connection*. The result is again a 2-groupoid $\mathcal{T}riv(\mathbb{G},\nabla)$ whose objects are the trivializations with connection compatible with $\nabla$.

## 5.2 Lemma 2.2.5: Bundle Gerbes act on Trivializations

We exhibit the 2-groupoid $\mathcal{T}riv(\mathbb{G})$ of trivializations of a bundle 2-gerbe $\mathbb{G}$ over $M$ as a module for the 2-category $\mathcal{G}rb(M)$ of bundle gerbes over $M$. The module structure is a strict 2-functor

$$\mathcal{T}riv(\mathbb{G}) \times \mathcal{G}rb(M) \longrightarrow \mathcal{T}riv(\mathbb{G}) \tag{5.2.1}$$

satisfying the usual axioms in a strict way.

We remark that all results of this section generalize analogous results for an action of principal U(1)-bundles over $M$ on trivializations of bundle (1-)gerbes, see [Wal07a], Theorem 2.5.4, and references therein.

If $\mathbb{T} = (\mathcal{S},\mathcal{A},\sigma)$ is a trivialization and $\mathcal{K}$ is a bundle gerbe over $M$, we obtain a new trivialization $\mathbb{T}.\mathcal{K}$ consisting of the bundle gerbe $\mathcal{S}\otimes\pi^*\mathcal{K}$ over $Y$. Since $\pi_1^*\pi^*\mathcal{K} = \pi_2^*\pi^*\mathcal{K}$, its isomorphism is simply

$$\mathcal{A}\otimes \mathrm{id} : \mathcal{P}\otimes\pi_2^*\mathcal{S}\otimes\pi_2^*\pi^*\mathcal{K} \longrightarrow \pi_1^*\mathcal{S}\otimes\pi_1^*\pi^*\mathcal{K}.$$

In the same way, its transformation is $\sigma\otimes\mathrm{id}$. If $\mathbb{B} = (\mathcal{B},\beta) : \mathbb{T} \longrightarrow \mathbb{T}'$ is a 1-morphism between trivializations and $\mathcal{J} : \mathcal{K} \longrightarrow \mathcal{K}'$ is an isomorphism between bundle gerbes, we obtain a new 1-morphism

$$\mathbb{B}.\mathcal{J} : \mathbb{T}.\mathcal{K} \longrightarrow \mathbb{T}'.\mathcal{K}'$$

consisting of the isomorphism $\mathcal{B}\otimes\pi^*\mathcal{J} : \mathcal{S}\otimes\pi^*\mathcal{K} \longrightarrow \mathcal{S}'\otimes\pi^*\mathcal{K}'$ and of the transformation $\beta\otimes\mathrm{id}$. Finally, if $\varphi : \mathbb{B} \Longrightarrow \mathbb{B}'$ is a 2-morphism between trivializations, and $\phi : \mathcal{J} \Longrightarrow \mathcal{J}'$ is a transformation between isomorphisms of bundle gerbes over $M$, we have a new 2-morphism

$$\varphi.\phi : \mathbb{B}.\mathcal{J} \Longrightarrow \mathbb{B}'.\mathcal{J}'$$

simply defined by $\varphi \otimes \pi^*\phi$. The compatibility condition for the transformation $\varphi \otimes \pi^*\phi$ is satisfied since $\phi$ drops out due to $\pi_1^*\pi^*\phi = \pi_2^*\pi^*\phi$ over $Y^{[2]}$.

Summarizing, the action of bundle gerbes on trivializations is a combination of the pullback $\pi^*$ and the tensor product of the monoidal 2-category of bundle gerbes. From this point of view, all the axioms of the action 2-functor (5.2.1) follow from those of the monoidal structure. It is also immediately clear that a genuine action on isomorphism classes is induced. It remains to show that this action is free and transitive.

To see that the action is free, assume that there exists a 1-morphism $\mathbb{T}.\mathcal{K} \longrightarrow \mathbb{T}$ for $\mathbb{T}$ a trivialization of $\mathbb{G}$ and $\mathcal{K}$ an isomorphism. This implies

$$\mathrm{DD}(\mathcal{S}) + \pi^*\mathrm{DD}(\mathcal{K}) = \mathrm{DD}(\mathcal{S}),$$

so that $\pi^*\mathrm{DD}(\mathcal{K}) = 0$. Since $\pi$ is a covering, $\mathrm{DD}(\mathcal{K}) = 0$. Thus, $\mathcal{K}$ is a trivial bundle gerbe up to isomorphism.

To see that the action is transitive we infer that bundle gerbes form a 2-stack over smooth manifolds. The gluing property of this 2-stack has been shown in [Ste00], Prop. 6.7. We also use the duality on the 2-groupoid of bundle gerbes (see Section 4.1).

Suppose $\mathbb{T}_1 = (\mathcal{S}_1, \mathcal{A}_1, \sigma_1)$ and $\mathbb{T}_2 = (\mathcal{S}_2, \mathcal{A}_2, \sigma_2)$ are trivializations of a bundle 2-gerbe $\mathbb{G}$. We will show that the bundle gerbe $\mathcal{G} := \mathcal{S}_1^* \otimes \mathcal{S}_2$ over $Y$ is a descent object for the 2-stack of bundle gerbes: there is an isomorphism $\mathcal{J} : \pi_2^*\mathcal{G} \longrightarrow \pi_1^*\mathcal{G}$ of bundle gerbes over $Y^{[2]}$ and a 2-isomorphism

$$\varphi : \pi_{12}^*\mathcal{J} \circ \pi_{23}^*\mathcal{J} \Longrightarrow \pi_{13}^*\mathcal{J}$$

over $Y^{[3]}$ which satisfies an associativity condition over $Y^{[4]}$. Then, the gluing property implies the existence of a bundle gerbe $\mathcal{K}$ over $M$, of an isomorphism $\mathcal{C} : \pi^*\mathcal{K} \longrightarrow \mathcal{G}$ and of a transformation

$$\gamma : \pi_1^*\mathcal{C} \Longrightarrow \mathcal{J} \circ \pi_2^*\mathcal{C}$$

such that the diagram

$$\begin{tikzcd}
\pi_{12}^*\mathcal{J} \circ \pi_2^*\mathcal{C} \arrow[r, Rightarrow, "\mathrm{id}\circ\pi_{23}^*\gamma"] & \pi_{12}^*\mathcal{J} \circ \pi_{23}^*\mathcal{J} \circ \pi_3^*\mathcal{C} \arrow[d, Rightarrow, "\varphi\circ\mathrm{id}"] \\
\pi_1^*\mathcal{C} \arrow[u, Rightarrow, "\pi_{12}^*\gamma"] \arrow[r, Rightarrow, "\pi_{13}^*\gamma"'] & \pi_{13}^*\mathcal{J} \circ \pi_3^*\mathcal{C}
\end{tikzcd} \tag{5.2.2}$$

is commutative. We will then finish the proof of the transitivity by showing that $(\mathcal{C}, \gamma)$ gives rise to a 1-morphism $\mathbb{T}_1.\mathcal{K} \longrightarrow \mathbb{T}_2$.

Let us first define the descent data $(\mathcal{J}, \varphi)$ for $\mathcal{G} = \mathcal{S}_1^* \otimes \mathcal{S}_2$. The isomorphism $\mathcal{J}$ is defined as the composition

$$\begin{tikzcd}
\pi_2(\mathcal{S}_1^* \otimes \mathcal{S}_2) \arrow[r, equal] & \pi_2^*\mathcal{S}_1^* \otimes \mathcal{I} \otimes \pi_2^*\mathcal{S}_2 \arrow[d, "\mathrm{id}\otimes\mathcal{D}_{\mathcal{P}}^{-1}\otimes\mathrm{id}"] & \\
& \pi_2^*\mathcal{S}_1^* \otimes \mathcal{P}^* \otimes \mathcal{P} \otimes \pi_2^*\mathcal{S}_2 \arrow[d, "\mathcal{A}_1^{*-1}\otimes\mathcal{A}_2"] & \\
& \pi_1^*\mathcal{S}_1^* \otimes \pi_1^*\mathcal{S}_2 \arrow[r, equal] & \pi_1^*(\mathcal{S}_1^* \otimes \mathcal{S}_2).
\end{tikzcd}$$

The transformation $\varphi$ is defined using the transformations $\sigma_1$ and $\sigma_2$, namely as the composition

$$\begin{array}{c}
\pi_{12}^*\mathcal{J} \circ \pi_{23}^*\mathcal{J} \Longrightarrow (\pi_{12}^*\mathcal{A}_1^{*-1} \otimes \pi_{12}^*\mathcal{A}_2) \circ (\pi_{23}^*\mathcal{A}_1^{*-1} \otimes \pi_{23}^*\mathcal{A}_2) \circ \mathcal{D}^{-1}_{\pi_{12}^*\mathcal{P}\otimes\pi_{23}^*\mathcal{P}} \\
\Big\Downarrow {\scriptstyle \sigma_1^{*-1}\otimes\sigma_2} \\
(\pi_{13}^*\mathcal{A}_1^{*-1} \otimes \pi_{13}^*\mathcal{A}_2) \circ (\mathcal{M}^{*-1} \otimes \mathcal{M}) \circ \mathcal{D}^{-1}_{\pi_{12}^*\mathcal{P}\otimes\pi_{23}^*\mathcal{P}} \\
\Big\Downarrow {\scriptstyle \text{naturality of } \mathcal{D} \text{ w.r.t } \mathcal{M}} \\
(\pi_{13}^*\mathcal{A}_1^{*-1} \otimes \pi_{13}^*\mathcal{A}_2) \circ \mathcal{D}^{-1}_{\pi_{13}^*\mathcal{P}} = \pi_{13}^*\mathcal{J}.
\end{array}$$

The associativity condition for $\varphi$ follows from the compatibility of $\sigma_1$ and $\sigma_2$ with the associator $\mu$ of $\mathbb{G}$ (see Figure 2 on page 40).

By the gluing axiom, we have now a bundle gerbe $\mathcal{K}$, an isomorphism $\mathcal{C}$, and a transformation $\gamma$ as claimed above. We define an isomorphism $\mathcal{B} : \mathcal{S}_1 \otimes \pi^*\mathcal{K} \longrightarrow \mathcal{S}_2$ as the composition

$$\mathcal{S}_1 \otimes \pi^*\mathcal{K} \xrightarrow{\mathrm{id}\otimes\mathcal{C}} \mathcal{S}_1 \otimes \mathcal{S}_1^* \otimes \mathcal{S}_2 \xrightarrow{\mathcal{D}_{\mathcal{S}_1^*}\otimes\mathrm{id}} \mathcal{I} \otimes \mathcal{S}_2 = \mathcal{S}_2.$$

By a similar manipulation one can produce a transformation

$$\begin{tikzcd}
\mathcal{P} \otimes \pi_2^*\mathcal{S}_1 \otimes \pi_2^*\pi^*\mathcal{K} \arrow[r, "\mathcal{A}_1\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\pi_2^*\mathcal{B}"'] & \pi_1^*\mathcal{S}_1 \otimes \pi_1^*\pi^*\mathcal{K} \arrow[d, "\pi_1^*\mathcal{B}"] \arrow[dl, Rightarrow, "\beta"'] \\
\mathcal{P} \otimes \pi_2^*\mathcal{S}_2 \arrow[r, "\mathcal{A}_2"'] & \pi_1^*\mathcal{S}_2
\end{tikzcd}$$

using $\gamma$. More precisely, $\beta$ is defined as

$$\begin{array}{c}
\pi_1^*\mathcal{B} \circ (\mathcal{A}_1 \otimes \mathrm{id}) = (\mathcal{D}_{\pi_1^*\mathcal{S}_1^*} \otimes \mathrm{id}) \circ (\mathcal{A}_1 \otimes \mathrm{id}^{\otimes 2}) \circ (\mathrm{id}^{\otimes 2} \otimes \pi_1^*\mathcal{C}) \\
\Big\Downarrow \scriptstyle \mathrm{id}\circ\mathrm{id}\circ(\mathrm{id}^{\otimes 2}\otimes\gamma) \\
(\mathcal{D}_{\pi_1^*\mathcal{S}_1^*} \otimes \mathrm{id}) \circ (\mathcal{A}_1 \otimes \mathrm{id}^{\otimes 2}) \circ (\mathrm{id}^{\otimes 2} \otimes \mathcal{J}) \circ (\mathrm{id}^{\otimes 2} \otimes \pi_2^*\mathcal{C}) \\
\Big\Downarrow \scriptstyle \text{Def. of } \mathcal{J} \\
(\mathcal{D}_{\pi_1^*\mathcal{S}_1^*} \otimes \mathrm{id}) \circ (\mathcal{A}_1 \otimes \mathcal{A}_1^{*-1} \otimes \mathrm{id}) \circ (\mathrm{id}^{\otimes 4} \otimes \mathcal{A}_2) \circ (\mathrm{id}^{\otimes 3} \otimes \mathcal{D}_{\mathcal{P}}^{-1} \otimes \mathrm{id}) \circ (\mathrm{id}^{\otimes 2} \otimes \pi_2^*\mathcal{C}) \\
\Big\Downarrow \scriptstyle \text{naturality of } \mathcal{D} \text{ applied to } \mathcal{A}_1 \\
(\mathcal{D}_{\pi_2^*\mathcal{S}_1} \otimes \mathcal{D}_{\mathcal{P}^*}) \circ (\mathrm{id}^{\otimes 4} \otimes \mathcal{A}_2) \circ (\mathrm{id}^{\otimes 3} \otimes \mathcal{D}_{\mathcal{P}}^{-1} \otimes \mathrm{id}) \circ (\mathrm{id}^{\otimes 2} \otimes \pi_2^*\mathcal{C}) \\
\Big\Downarrow \scriptstyle \text{compatibility between } \otimes \text{ and } \circ \\
(\mathcal{D}_{\mathcal{P}^*} \otimes \mathrm{id}) \circ (\mathrm{id}^{\otimes 2} \otimes \mathcal{A}_2) \circ (\mathrm{id}^{\otimes 3} \otimes \mathcal{D}_{\mathcal{P}}^{-1} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathcal{D}_{\pi_2^*\mathcal{S}_1^*} \otimes \mathrm{id}) \circ (\mathrm{id}^{\otimes 2} \otimes \pi_2^*\mathcal{C}) \\
\Big\Downarrow \scriptstyle \text{zigzag for } \mathcal{D}_{\mathcal{P}} \\
\mathcal{A}_2 \circ (\mathrm{id} \otimes \mathcal{D}_{\pi_2^*\mathcal{S}_1^*} \otimes \mathrm{id}) \circ (\mathrm{id}^{\otimes 2} \otimes \pi_2^*\mathcal{C}) = \mathcal{A}_2 \circ (\mathrm{id} \otimes \pi_2^*\mathcal{B}).
\end{array}$$

Finally, one can deduce from the commutativity of (5.2.2) and the definition of the transformation $\varphi$ that $\beta$ is compatible with the transformations $\sigma_1$ and $\sigma_2$ in the sense of Figure 3. Hence, $(\mathcal{B}, \beta)$ is a 1-morphism from $\mathbb{T}_1.\mathcal{K}$ to $\mathbb{T}_2$.

**Remark 5.2.1.** Since we have only used abstract properties of the 2-stack of bundle gerbes, the action and its properties generalize straightforwardly to the 2-stack of bundle gerbes *with connection*.

## 5.3 Lemma 3.2.2: Existence of Trivializations with Connection

We show that every bundle 2-gerbe $\mathbb{G}$ with connection and vanishing characteristic class admits a trivialization with compatible connection. For the proof we use the fact that bundle 2-gerbes are classified up to isomorphism by degree four differential cohomology $\hat{\mathrm{H}}^4(M, \mathbb{Z})$. This cohomology group fits into the exact sequence

$$\Omega^3(M) \longrightarrow \hat{\mathrm{H}}^4(M, \mathbb{Z}) \xrightarrow{\mathrm{CC}} \mathrm{H}^4(M, \mathbb{Z}) \longrightarrow 0. \tag{5.3.1}$$

Suppose $\mathbb{G}$ is a bundle 2-gerbe with connection $\nabla$ and with vanishing characteristic class $\mathrm{CC}(\mathbb{G})$. By exactness of (5.3.1), it is isomorphic to a certain bundle 2-gerbe $\mathbb{I}$ with connection $\nabla_H$ defined by a 3-form $H \in \Omega^3(M)$. We

will show that such an isomorphism is precisely a trivialization of $\mathbb{G}$ with connection compatible with $\nabla$. This proves Lemma 3.2.2.

Let us first describe the bundle 2-gerbe $\mathbb{I}$ and the connection $\nabla_H$ which is associated to any 3-form $H$ on $M$. The covering of $\mathbb{I}$ is the identity $\mathrm{id} : M \longrightarrow M$ whose fibre products we can identify with $M$ itself. Its bundle gerbe is the trivial bundle gerbe $\mathcal{I}$, which is the tensor unit of the monoidal 2-category $\mathcal{G}rb(M)$. Its product is the identity $\mathrm{id} : \mathcal{I} \otimes \mathcal{I} \longrightarrow \mathcal{I}$, and its associator is also the identity. The only non-trivial information is the connection $\nabla_H$. It consists simply of the 3-form $H$; the bundle gerbe $\mathcal{I}$ and the product of $\mathbb{I}$ carry trivial connections.

Next we give a brief definition of an isomorphism between two bundle 2-gerbes $\mathbb{G}_1$ and $\mathbb{G}_2$. It is a straightforward generalization of the notion of an isomorphism between bundle gerbes (see Definition 4.1.2). An *isomorphism* $\mathbb{A} : \mathbb{G}_1 \longrightarrow \mathbb{G}_2$ consists of a bundle gerbe $\mathcal{S}$ over a covering $\zeta : Z \longrightarrow Y_1 \times_M Y_2$, an isomorphism

$$\mathcal{A} : \mathcal{P}_1 \otimes \zeta_2^*\mathcal{S} \longrightarrow \zeta_1^*\mathcal{S} \otimes \mathcal{P}_2$$

between bundle gerbes over $Z \times_M Z$, and a transformation $\sigma$ which expresses the compatibility between $\mathcal{A}$ and the products $\mathcal{M}_1$ and $\mathcal{M}_2$. This transformation has to satisfy an evident coherence condition involving the associators $\mu_1$ and $\mu_2$ of the two bundle 2-gerbes.

If the bundle 2-gerbes $\mathbb{G}_1$ and $\mathbb{G}_2$ are equipped with connections, we say that a *compatible connection* on the isomorphism $\mathbb{A}$ is a connection on the bundle gerbe $\mathcal{S}$ of curvature

$$\mathrm{curv}(S) = \pi_2^*B_2 - \pi_1^*B_1,$$

and a compatible connection on the isomorphism $\mathcal{A}$ such that $\sigma$ is connection-preserving.

The claimed relation to differential cohomology (realized by Deligne cohomology) is established by Prop. 4.2 in [Joh02]: there is a canonical bijection

$$\left\{ \begin{array}{c} \text{Bundle 2-gerbes over } M \text{ with} \\ \text{connection, up to isomorphisms} \\ \text{with compatible connection} \end{array} \right\} \cong \hat{\mathrm{H}}^4(M, \mathbb{Z}).$$

Comparing the definition of an isomorphism between bundle 2-gerbes and Definition 2.2.1 of a trivialization makes it obvious that a trivialization $\mathbb{T}$ of $\mathbb{G}$ is the same thing as an isomorphism $\mathbb{T} : \mathbb{G} \longrightarrow \mathbb{I}$. This coincidence generalizes to a setup with compatible connections: there is a bijection

$$\left\{ \begin{array}{c} \text{Isomorphisms } \mathbb{T} : \mathbb{G} \longrightarrow \mathbb{I} \text{ with} \\ \text{connection compatible with } \nabla \\ \text{and } \nabla_H \text{ for some 3-form } H \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Trivializations of } \mathbb{G} \text{ with} \\ \text{connection compatible with } \nabla \end{array} \right\}.$$

for every bundle 2-gerbe $\mathbb{G}$ and any connection $\nabla$ on $\mathbb{G}$.

## 5.4 Lemma 3.3.3: Connections on Trivializations pull back

Let $\mathbb{G}$ be a bundle 2-gerbe with connection, and let $\mathbb{T} = (\mathcal{S}, \mathcal{A}, \sigma)$ and $\mathbb{T}' = (\mathcal{S}', \mathcal{A}', \sigma')$ be trivializations of $\mathbb{G}$. We prove that one can pull back a compatible connection $\blacktriangledown'$ on $\mathbb{T}'$ along any 1-morphism $\mathbb{B} : \mathbb{T} \longrightarrow \mathbb{T}'$ to a compatible connection on $\mathbb{T}$.

We recall that a compatible connection on $\mathbb{T}'$ is a pair $\blacktriangledown' = (\nabla', \omega')$ with $\nabla'$ a connection on $\mathcal{S}'$ and $\omega'$ a connection on $\mathcal{A}'$, such that $\sigma'$ is connection-preserving.

By Lemma 4.2.4, there exists a connection $\kappa$ on the isomorphism $\mathcal{B} : \mathcal{S} \longrightarrow \mathcal{S}'$ and a connection on $\mathcal{S}$ such that $\kappa$ is compatible. Next we need a connection on the isomorphism $\mathcal{A}$. We look at the transformation

$$\mathcal{A} \xRightarrow{\rho_{\mathcal{A}}^{-1}} \mathrm{id} \circ \mathcal{A} \xRightarrow{\pi_1^* i_r \otimes \mathrm{id}} \pi_1^*\mathcal{B} \circ \pi_1^*\mathcal{B}^{-1} \circ \mathcal{A} \xRightarrow{\mathrm{id} \circ \beta} \pi_1^*\mathcal{B}^{-1} \circ \mathcal{A}' \circ \pi_2^*\mathcal{B}. \quad (5.4.1)$$

Here we have used the transformation $\rho_{\mathcal{A}} : \mathrm{id} \circ \mathcal{A} \Longrightarrow \mathcal{A}$ which belongs to the structure of the 2-groupoid of bundle gerbes (see [Wal07b], Sec 1.2), and the canonical transformation $i_r : \mathrm{id} \Longrightarrow \mathcal{B} \circ \mathcal{B}^{-1}$ which expresses the invertibility of the isomorphism $\mathcal{B}$ (see [Wal07b], Sec. 1.3). Notice that the target isomorphism of (5.4.1) is equipped with a compatible connection. Thus, by Lemma 4.2.5, this connection pulls back to a compatible connection on $\mathcal{A}$. Furthermore, since the transformations (5.4.1), $\rho_{\mathcal{A}}$ and $i_r$ are connection-preserving, also the transformation $\mathrm{id} \circ \beta$ is connection-preserving. This implies in turn that $\beta$ itself is connection-preserving.

It remains to check that the transformation $\sigma$ preserves connections. In order to see this, consider the commutative diagram of Figure 3, which expresses the compatibility between $\sigma$, $\sigma'$ and $\beta$. Since all transformations that appear in this diagram are invertible, we can rearrange it as an equation

$$\mathrm{id} \circ \sigma = (\pi_{13}^*\beta \circ \mathrm{id})^{-1} \bullet (\sigma' \circ \mathrm{id}) \bullet (\mathrm{id} \circ \pi_{23}^*\beta) \bullet (\pi_{12}^*\beta \circ \mathrm{id})^{-1},$$

where $\bullet$ denotes the vertical composition of transformations. Now, since the right hand side of this equation is a connection-preserving transformation, also $\mathrm{id} \circ \sigma$ is connection-preserving. Just like above, it follows that $\sigma$ is connection-preserving.

## 5.5 Lemma 3.3.7 (a): Well-definedness of the Action on compatible Connections

We prove that the action of the vector space $V_{\mathbb{T}}$ on the connections on a trivialization $\mathbb{T}$ of a bundle 2-gerbe $\mathbb{G}$ with connection takes compatible connections to compatible connections. First we have to fix some notation.

The bundle 2-gerbe $\mathbb{G}$ has a covering $\pi : Y \longrightarrow M$ and a bundle gerbe $\mathcal{P}$ over $Y^{[2]}$. The bundle gerbe $\mathcal{P}$ in turn has a covering $\chi : X \longrightarrow Y^{[2]}$. The trivialization $\mathbb{T}$ has a bundle gerbe $\mathcal{S}$ over $Y$, an isomorphism $\mathcal{A}$ over $Y^{[2]}$ and a transformation $\sigma$. Its bundle gerbe $\mathcal{S}$ has a covering $\omega : W \longrightarrow Y$. Expanding the definitions of tensor products and isomorphisms between bundle gerbes, the isomorphism $\mathcal{A} : \mathcal{P} \otimes \pi_2^*\mathcal{S} \longrightarrow \pi_1^*\mathcal{S}$ comes with a covering

$$\zeta : Z \longrightarrow X \times_{Y^{[2]}} (W \times_M W).$$

The projections $x : Z \longrightarrow X$ and $p : Z \longrightarrow W \times_M W$ are again coverings. By construction, there is a commutative diagram

$$\begin{tikzcd}
 & Z \arrow[d, "p"] & \\
 & W \times_M W \arrow[dl, "p_2"'] \arrow[dr, "p_1"] \arrow[dd, "\omega\times\omega"] & \\
W \arrow[dd, "\omega"'] & & W \arrow[dd, "\omega"] \\
 & Y^{[2]} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
Y \arrow[dr, "\pi"'] & & Y \arrow[dl, "\pi"] \\
 & M &
\end{tikzcd} \tag{5.5.1}$$

We keep a connection on $\mathbb{G}$ fixed, and assume a compatible connection $\blacktriangledown = (C, \omega, \kappa)$ on $\mathbb{T}$, where $(C, \omega)$ is a connection on $\mathcal{S}$. Let $(\psi, \rho, \varphi) \in V_{\mathbb{T}}$ represent an element in the vector space that acts on the set of connections of $\mathbb{T}$. Its action on $\blacktriangledown$ has been defined in Section 3.3 to result in $\blacktriangledown' = (C', \omega', \kappa')$ with

$$\begin{aligned}
C' &= C + \mathrm{d}\varphi - \omega^*(\mathrm{d}\rho - \pi^*\psi) \\
\omega' &= \omega + \delta_\omega \varphi \\
\kappa' &= \kappa + \epsilon \quad \text{with} \quad \epsilon := p^*(\delta_{\pi\circ\omega}(\varphi - \omega^*\rho)).
\end{aligned} \tag{5.5.2}$$

Here we have used the notation $\delta_{\pi\circ\omega}$, $\delta_\omega$ for the alternating sum over pullbacks along a surjective submersion, as explained in Lemma 4.2.2.

Let us first check that $\kappa'$ is a compatible connection on $\mathcal{A}$. The first condition is the equation

$$\mathrm{curv}(\kappa') = p^*p_2^*C' - (x^*C_{\mathcal{P}} + p^*p_1^*C') \tag{5.5.3}$$

of 2-forms over $Z$, where $C_{\mathcal{P}}$ is the 2-form of the connection on $\mathcal{P}$. This equation can be verified using the fact that $\kappa$ was assumed to be compatible, and using the commutative diagram (5.5.1) to sort out the various pullbacks. The second condition for $\kappa'$ is that the isomorphism

$$\alpha : (x^*P_{\mathcal{P}} \otimes w_1^*P_{\mathcal{S}}) \otimes \zeta_2^*Q \longrightarrow \zeta_1^*Q \otimes w_2^*P_{\mathcal{S}}$$

of principal U(1)-bundles over $Z \times_{Y^{[2]}} Z$ is connection-preserving. Here, $\zeta_1, \zeta_2 : Z \times_{Y^{[2]}} Z \longrightarrow Z$ denote the two projections, $x : Z \times_{Y^{[2]}} Z \longrightarrow X \times_{Y^{[2]}} X$ is just the map $x$ from above on each factor, and $w_i : Z \times_{Y^{[2]}} Z \longrightarrow W \times_Y W$ is the map $p_i \circ p : Z \longrightarrow W$ on each factor. There is a commutative diagram exploiting the various relations between these maps ($k = 1, 2$):

$$\begin{tikzcd}
& & Z \times_{Y^{[2]}} Z \arrow[dll, "\zeta_1"'] \arrow[d, "w_k"] \arrow[drr, "\zeta_2"] & & \\
Z \arrow[dr, "p_k"'] & & W \times_Y W \arrow[dl, "\omega_1"'] \arrow[dr, "\omega_2"] & & Z \arrow[dl, "p_k"] \\
& W \arrow[dr, "\omega"'] & & W \arrow[dl, "\omega"] & \\
& & Y & &
\end{tikzcd} \tag{5.5.4}$$

To check that $\alpha$ is connection-preserving one verifies that the connections on both sides change by the same 1-form on $Z \times_{Y^{[2]}} Z$ under the action (5.5.2). On the left hand side, this is $w_1^*\delta_\omega\varphi + \zeta_2^*\epsilon$. On the right hand side, it is $\zeta_1^*\epsilon + w_2^*\delta_\omega\varphi$. Using diagram (5.5.4) it is straightforward to check that these 1-forms coincide.

The remaining check is that the transformation $\sigma$ is still connection-preserving. We note that $\sigma$ is some isomorphism of principal U(1)-bundles over a smooth manifold $V$. There are projections $k_s$ and $k_t$ into the covering spaces of the source isomorphism $\pi_{12}^*\mathcal{A} \circ (\mathrm{id}_{\pi_{12}^*\mathcal{P}} \otimes \pi_{23}^*\mathcal{A})$ and the target isomorphism $\pi_{13}^*\mathcal{A} \circ (\mathcal{M} \otimes \mathrm{id}_{\pi_3^*\mathcal{S}})$. We can safely ignore the contributions of the isomorphism $\mathcal{M}$ and of the identity isomorphism $\mathrm{id}_{\pi_{12}^*\mathcal{P}}$ in the following discussion, since the connections on $\mathcal{M}$ and on $\mathcal{P}$ did not change under our action. What we must not ignore is the identity isomorphism $\mathrm{id}_{\pi_3^*\mathcal{S}}$: its connection changes with the connection on $\mathcal{S}$!

After these premises, the two projections are $k_s : V \longrightarrow Z \times_W Z$ and $k_t : V \longrightarrow (W \times_Y W) \times_W Z$, with $Z$ the covering space of the isomorphism $\mathcal{A}$ and $W \times_Y W$ the covering space of the identity isomorphism. We claim that the following diagrams are commutative by construction:

$$
\begin{tikzcd}
 & V \arrow[dl, "k_s"'] \arrow[dr, "k_t"] & & \\
Z\times_W Z \arrow[d, "p\circ\zeta_1"'] & & (W\times_Y W)\times_W Z \arrow[d, "\mathrm{pr}_1"] & \\
W\times_M W \arrow[dr, "\omega_1"'] & & W\times_Y W \arrow[dl, "\omega_1"'] \arrow[dr, "\omega_2"] & \\
 & W \arrow[dr, "\omega"'] & & W \arrow[dl, "\omega"] \\
 & & Y &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & V \arrow[d, "k_s"] & \\
 & Z\times_W Z \arrow[dl, "\zeta_1"'] \arrow[dr, "\zeta_2"] & \\
Z \arrow[d, "p"'] & & Z \arrow[d, "p"] \\
W\times_M W \arrow[dr, "\omega_2"'] & & W\times_M W \arrow[dl, "\omega_1"] \\
 & W &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & V \arrow[dl, "k_s"'] \arrow[dr, "k_t"] & \\
Z\times_W Z \arrow[d, "\zeta_2"'] & & (W\times_Y W)\times_W Z \arrow[d, "\mathrm{pr}_2"] \\
Z \arrow[d, "p"'] & & Z \arrow[d, "p"] \\
W\times_M W \arrow[dr, "\omega_2"'] & & W\times_M W \arrow[dl, "\omega_2"] \\
 & W &
\end{tikzcd}
$$

We compute the changes in the connections on the target and on the source isomorphism of $\sigma$. These are, respectively, the 1-forms

$$k_t^*(\mathrm{pr}_1^*\delta\varphi + \mathrm{pr}_2^*\epsilon) \quad \text{and} \quad k_s^*(\zeta_1^*\epsilon + \zeta_2^*\epsilon) \tag{5.5.5}$$

over $V$. Using the diagrams above it is a straightforward calculation to check that these 1-forms coincide. Thus, $\sigma$ is a connection-preserving transformation.

Summarizing, we have shown that the action of an element of $V_{\mathbb{T}}$ on a compatible connection $\blacktriangledown$ on $\mathbb{T}$ is again a compatible connection $\blacktriangledown'$.

## 5.6 Lemma 3.3.7 (b): The Action on compatible Connections is free and transitive

We prove that the action of the vector space $V_{\mathbb{T}}$ on the compatible connections of a bundle 2-gerbe $\mathbb{G}$ with connection is free and transitive.

Showing that the action is free is the easy part. We assume that an element $(\psi, \rho, \varphi) \in V_{\mathbb{T}}$ acts trivially on a connection $\blacktriangledown = (C, \omega, \kappa)$ on a trivialization $\mathbb{T} = (\mathcal{S}, \mathcal{A}, \sigma)$. That means that the 1-form $\epsilon = p^*\delta_{\pi\circ\omega}(\varphi - \omega^*\rho)$ vanishes and that $(\eta, \varphi) \in V_{\mathcal{S}}$ with $\eta = \mathrm{d}\rho - \pi^*\psi$ vanishes in $V_{\mathcal{S}}$.

Since $p$ is a covering, the vanishing of $\epsilon$ implies that already $\delta_{\pi\circ\omega}(\varphi - \omega^*\rho) = 0$. By Lemma 4.2.2, this implies the existence of a 1-form $\chi \in \Omega^1(M)$ such that (I) $\omega^*\pi^*\chi = \omega^*\rho - \varphi$. The vanishing of $(\eta, \varphi)$ implies the existence

of a 1-form $\nu \in \Omega^1(Y)$ such that (II) $\eta = \mathrm{d}\nu$ and (III) $\varphi = \omega^*\nu$. (I) and (III) imply (IV) $\rho = \pi^*\chi + \nu$. (II) and (IV) imply (V) $\mathrm{d}\chi = \psi$. Equations (V), (IV) and (III) show that $(\psi, \rho, \varphi)$ lies in the subspace $U$ we divide out in Proposition 3.3.4.

Now we prove that the action is transitive. We assume that $\blacktriangledown = (C, \varphi, \kappa)$ and $\blacktriangledown' = (C', \varphi', \kappa')$ are two compatible connections on a trivialization $\mathbb{T}$. From Lemma 3.3.5 and Lemma 3.3.6 we obtain $\epsilon \in \Omega^1(Z)$, $\varphi \in \Omega^1(W)$ and $\eta \in \Omega^2(Y)$ such that

$$C' = C + \mathrm{d}\varphi - \omega^*\eta \quad , \quad \omega' = \omega + \delta_\omega\varphi \quad \text{and} \quad \kappa' = \kappa + \epsilon.$$

First we consider the 1-form

$$\tilde{\rho} := \epsilon - p_2^*\varphi + p_1^*\varphi \in \Omega^1(Z)$$

with $p_k : Z \longrightarrow W$ the projections from Section 5.5. We denote the evident projection to the base space of $\mathcal{A}$ by $\ell : Z \longrightarrow Y^{[2]}$. Using the identity

$$\zeta_2^*\epsilon = \zeta_1^*\epsilon \ + w_2^*\delta_\omega\varphi - w_1^*\delta_\omega\varphi$$

that we have derived in Section 5.5 it is straightforward to check that

$$\delta_\ell\tilde{\rho} = \zeta_2^*\tilde{\rho} - \zeta_1^*\tilde{\rho} = 0,$$

so that by Lemma 4.2.2 there exists a 1-form $\rho' \in \Omega^1(Y^{[2]})$ such that $\ell^*\rho' = \tilde{\rho}$.

Now we denote by $V$ the covering space of the transformation $\sigma$, and we denote by $k : V \longrightarrow Y^{[3]}$ the evident projection to the base space of the involved bundle gerbes. Note that the following diagrams are commutative:

$$\begin{tikzcd} V \arrow[r, "\zeta_2\circ k_s"] \arrow[d, "k"'] & Z \arrow[d, "\ell"] \\ Y^{[3]} \arrow[r, "\pi_{21}"'] & Y^{[2]} \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "\zeta_1\circ k_s"] \arrow[d, "k"'] & Z \arrow[d, "\ell"] \\ Y^{[3]} \arrow[r, "\pi_{32}"'] & Y^{[2]} \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "\mathrm{pr}_2\circ k_t"] \arrow[d, "k"'] & Z \arrow[d, "\ell"] \\ Y^{[3]} \arrow[r, "\pi_{31}"'] & Y^{[2]} \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "\mathrm{pr}_2\circ k_t"] \arrow[d, "\mathrm{pr}_1\circ k_t"'] & Z \arrow[d, "p_1"] \\ W\times_Y W \arrow[r, "\omega_2"'] & W. \end{tikzcd}$$

Using these diagrams and the coincidence of the 1-forms (5.5.5), one readily verifies

$$k^*(\pi_{21}^*\rho' - \pi_{31}^*\rho' + \pi_{32}^*\rho') = k_s^*\zeta_2^*\tilde{\rho} - k_t^*\mathrm{pr}_2^*\tilde{\rho} + k_s^*\zeta_1^*\tilde{\rho} = 0,$$

so that again by Lemma 4.2.2 there exists a 1-form $\rho \in \Omega^1(Y)$ such that $\delta_\pi\rho = \rho'$.

Finally we consider $\psi' := \mathrm{d}\rho - \eta \in \Omega^2(Y)$. We compute

$$\ell^*(\pi_2^*\psi' - \pi_1^*\psi') = 0,$$

using the remaining condition (5.5.3). Thus, we find a 1-form $\psi \in \Omega^2(M)$ such that $\pi^*\psi = \psi'$. Tracing all definitions back, we see that acting with $(\psi, \rho, \epsilon)$ on $\blacktriangledown$ we obtain $\blacktriangledown'$.

# Figures

$$
\begin{tikzcd}
 & & * \arrow[dll, Rightarrow, "\mathrm{id} \circ (\pi_{1234}^*\mu \otimes \mathrm{id})"'] \arrow[drr, Rightarrow, "\pi_{1345}^*\mu \circ \mathrm{id}"] & & \\
* \arrow[dr, Rightarrow, "\pi_{1245}^*\mu \circ \mathrm{id}"'] & & & & * \arrow[dl, Rightarrow, "\pi_{1235}^*\mu \circ \mathrm{id}"] \\
 & * \arrow[rr, Rightarrow, "\mathrm{id} \circ (\mathrm{id} \otimes \pi_{2345}^*\mu)"'] & & * &
\end{tikzcd}
$$

**Figure 1:** The pentagon axiom for the associator $\mu$ of a bundle 2-gerbe. It is an equation of two transformations between isomorphisms of bundle gerbes over $Y^{[5]}$.

$$
\begin{tikzcd}
 & \pi_{14}^*\mathcal{A} \circ (\pi_{134}^*\mathcal{M} \otimes \mathrm{id}) \circ (\pi_{123}^*\mathcal{M} \otimes \mathrm{id} \otimes \mathrm{id}) \arrow[dl, Rightarrow, "\mathrm{id} \circ (\mu \otimes \mathrm{id})"'] \arrow[dr, Rightarrow, "\pi_{134}^*\sigma"] & \\
\pi_{14}^*\mathcal{A} \circ (\pi_{124}^*\mathcal{M} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \pi_{234}^*\mathcal{M} \otimes \mathrm{id}) \arrow[d, Rightarrow, "\pi_{124}^*\sigma \circ \mathrm{id}"'] & & \pi_{13}^*\mathcal{A} \circ (\pi_{123}^*\mathcal{M} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \pi_{34}^*\mathcal{A}) \arrow[d, Rightarrow, "\pi_{123}^*\sigma \circ \mathrm{id}"] \\
\pi_{12}^*\mathcal{A} \circ (\mathrm{id} \otimes \pi_{24}^*\mathcal{A}) \circ (\mathrm{id} \otimes \pi_{234}^*\mathcal{M} \otimes \mathrm{id}) \arrow[rr, Rightarrow, bend right, "\mathrm{id} \circ (\mathrm{id} \otimes \pi_{234}^*\sigma)"'] & & \pi_{12}^*\mathcal{A} \circ (\mathrm{id} \otimes \pi_{23}^*\mathcal{A}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \pi_{34}^*\mathcal{A})
\end{tikzcd}
$$

**Figure 2:** The compatibility between the associator $\mu$ of a bundle 2-gerbe and the transformation $\sigma$ of a trivialization.

$$\begin{tikzcd}
& \pi_{12}^*\mathcal{A}' \circ (\mathrm{id} \otimes \pi_2^*\mathcal{B}) \circ (\mathrm{id} \otimes \pi_{23}^*\mathcal{A}) \arrow[dl, Rightarrow, "\pi_{12}^*\beta \circ \mathrm{id}"'] \arrow[dr, Rightarrow, "\mathrm{id} \circ \pi_{23}^*\beta"] & \\
\pi_1^*\mathcal{B} \circ \pi_{12}^*\mathcal{A}' \circ (\mathrm{id} \otimes \pi_{23}^*\mathcal{A}) \arrow[d, Rightarrow, "\mathrm{id} \circ \sigma"'] & & \pi_{12}^*\mathcal{A}' \circ (\mathrm{id} \otimes \pi_{23}^*\mathcal{A}') \circ (\mathrm{id} \otimes \mathrm{id} \otimes \pi_3^*\mathcal{B}) \arrow[d, Rightarrow, "\sigma' \circ \mathrm{id}"] \\
\pi_1^*\mathcal{B} \circ (\mathrm{id} \otimes \pi_{13}^*\mathcal{A}) \circ (\mathcal{M} \otimes \mathrm{id}) \arrow[rr, Rightarrow, "\pi_{13}^*\beta \circ \mathrm{id}"'] & & \pi_{13}^*\mathcal{A}' \circ (\mathcal{M} \otimes \pi_3^*\mathcal{B})
\end{tikzcd}$$

**Figure 3:** The compatibility between the transformations $\sigma$ and $\sigma'$ of two trivializations $\mathbb{T}$ and $\mathbb{T}'$ and the transformation $\beta$ of a 1-morphism $\mathbb{B} = (\mathcal{B}, \beta)$ between $\mathbb{T}$ and $\mathbb{T}'$.

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